Matching Problems

A 3/2-approximation algorithm for the student-project allocation problem with ties

Frances Cooper

Supervisor: Dr David Manlove

Outline

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- Algorithms

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- Algorithms
- Matching problems

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- Matching problems
- Stable matchings

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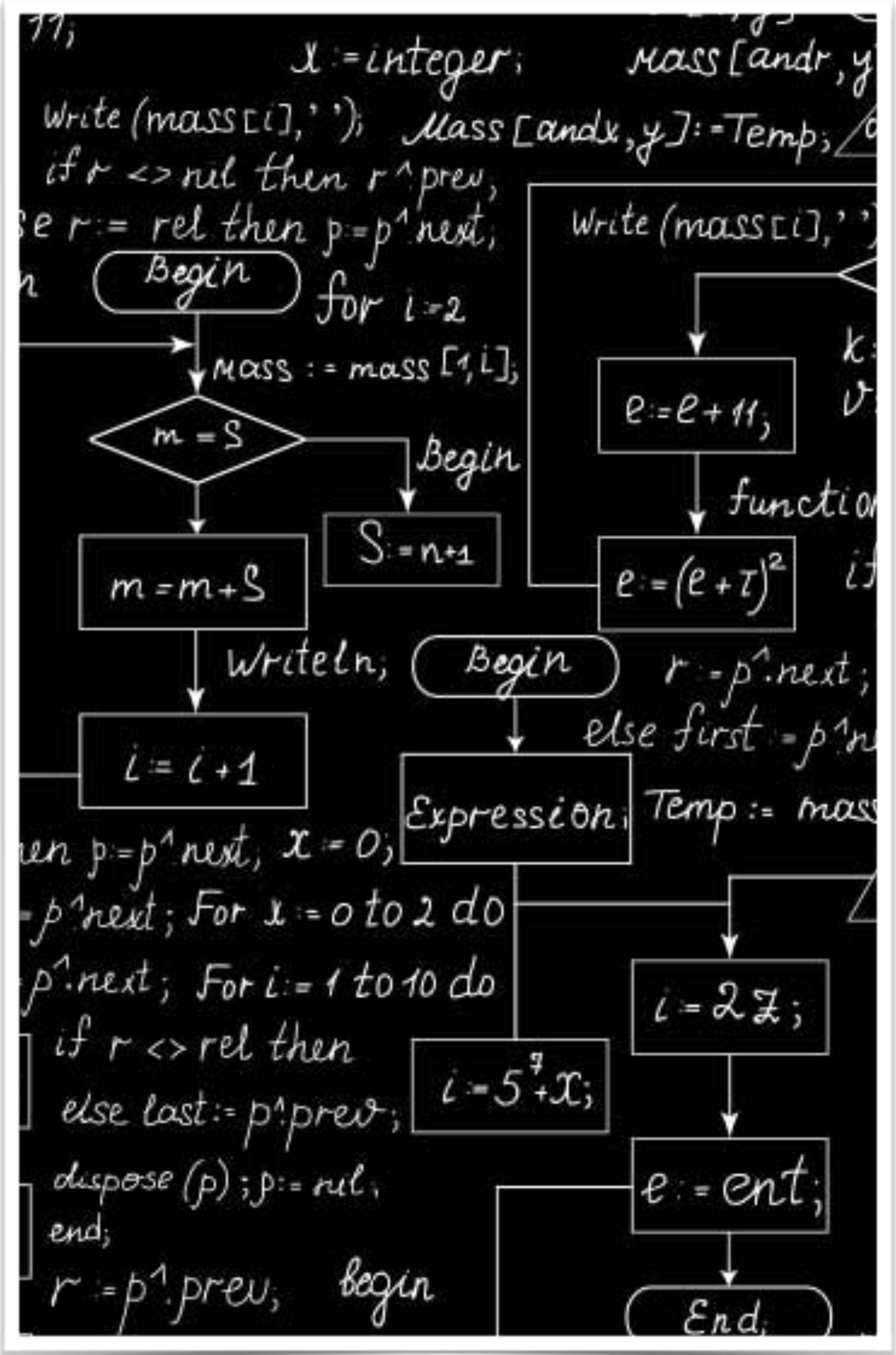
- Algorithms
- Matching problems
- Stable matchings
- Finding maximum stable matchings

Algorithms



x := integer;
mass [andr, y
Write (mass[i], ' ');
Mass [andx, y] := Temp;
if r <> nil then r^prev;
e r := rel then p := p^next;
Write (mass[i], ' '
Begin
for i := 2
mass : = mass [1, i];
m = S
Begin
S := n+1
e := e + 11;
function
m = m+S
e := (e + τ)²
Writeln;
Begin
r := p^next;
else first := p^n
i = i + 1
Expression;
Temp := mass
en p := p^next; x := 0;
p^next; For x := 0 to 2 do
p^next; For i := 1 to 10 do
if r <> rel then
i := 2Z;
else last := p^prev;
i := 5⁷+x;
dispose (p); p := nil;
end;
e := ent;
r := p^prev;
begin
End;

Algorithms

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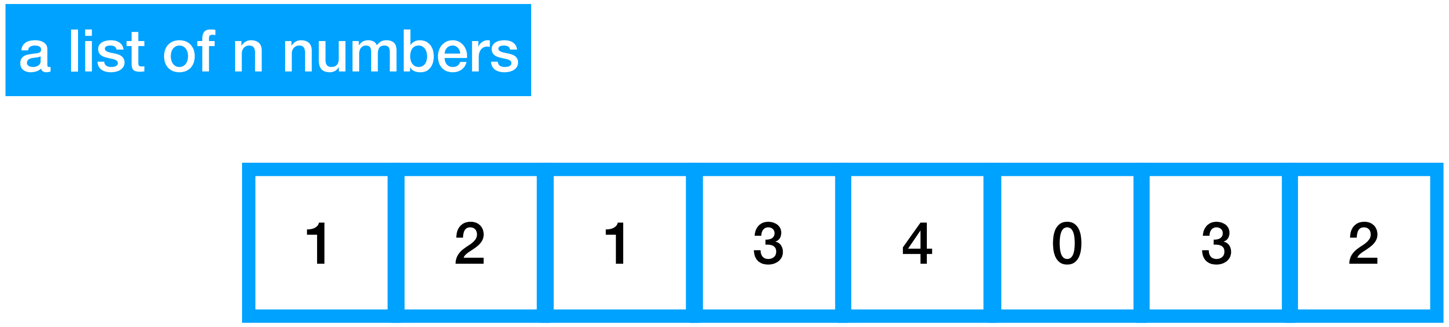

a list of n numbers

1	2	1	3	4	0	3	2

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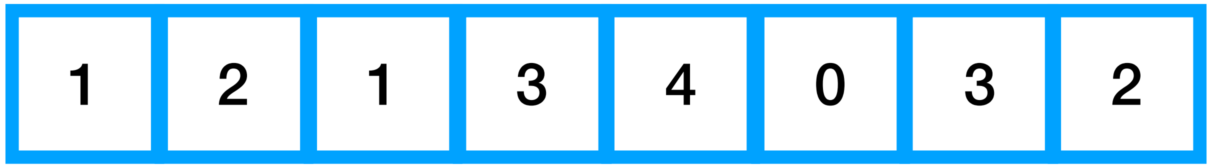
a list of n numbers

task: double each number



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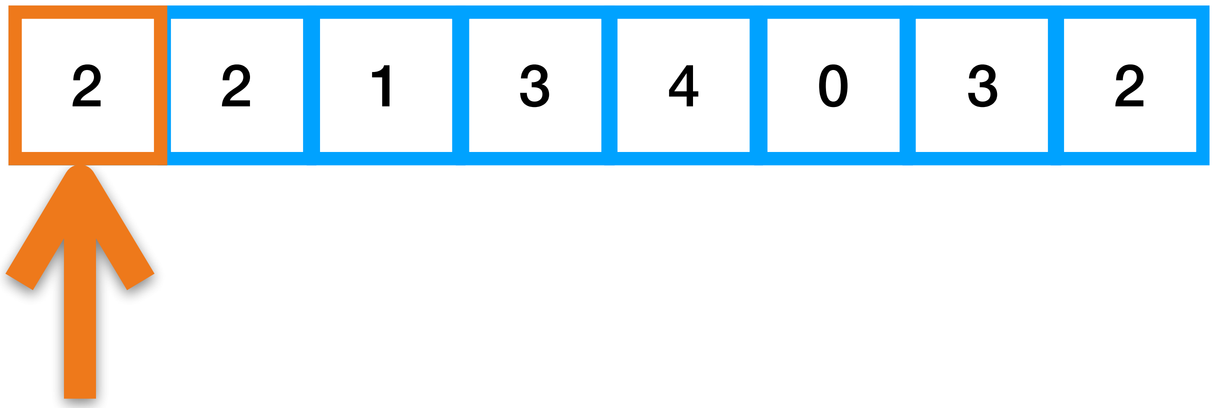

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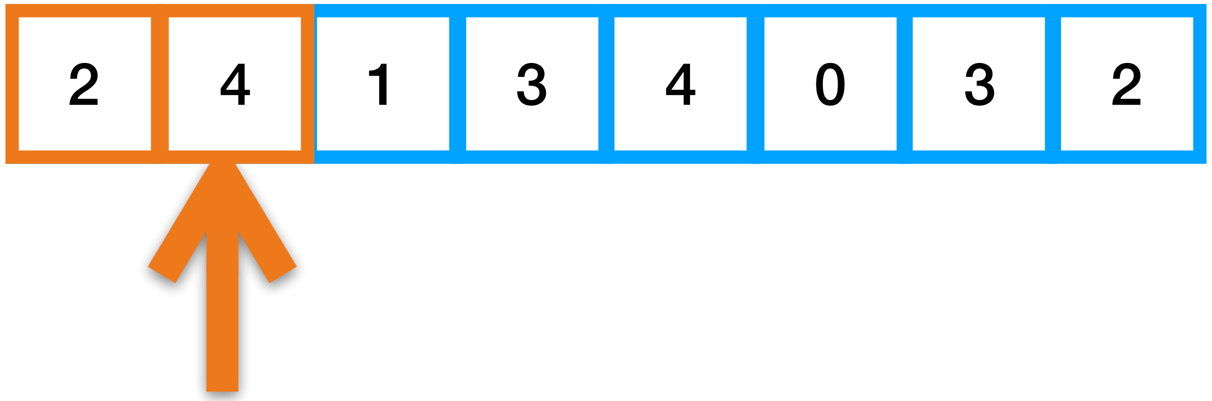

2	4	1	3	4	0	3	2

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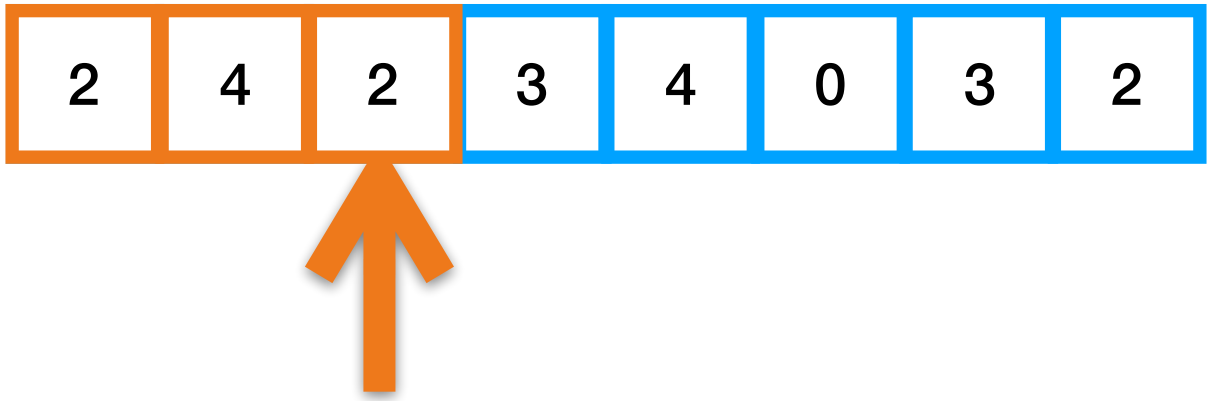

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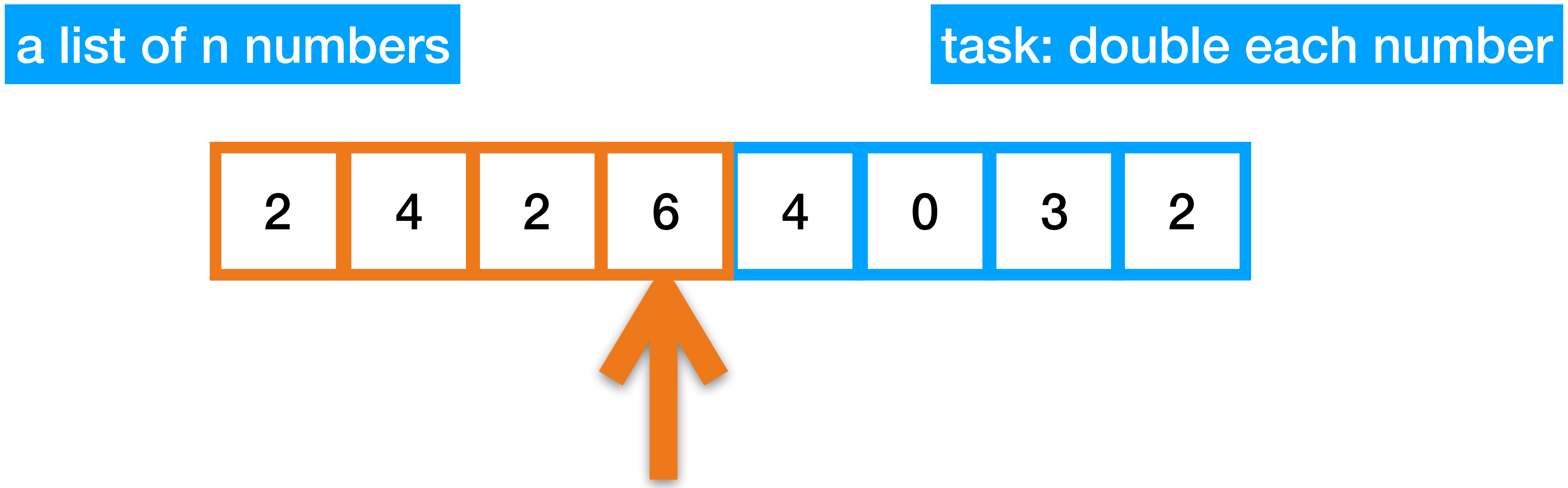

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2	4	2	6	4	0	3	2

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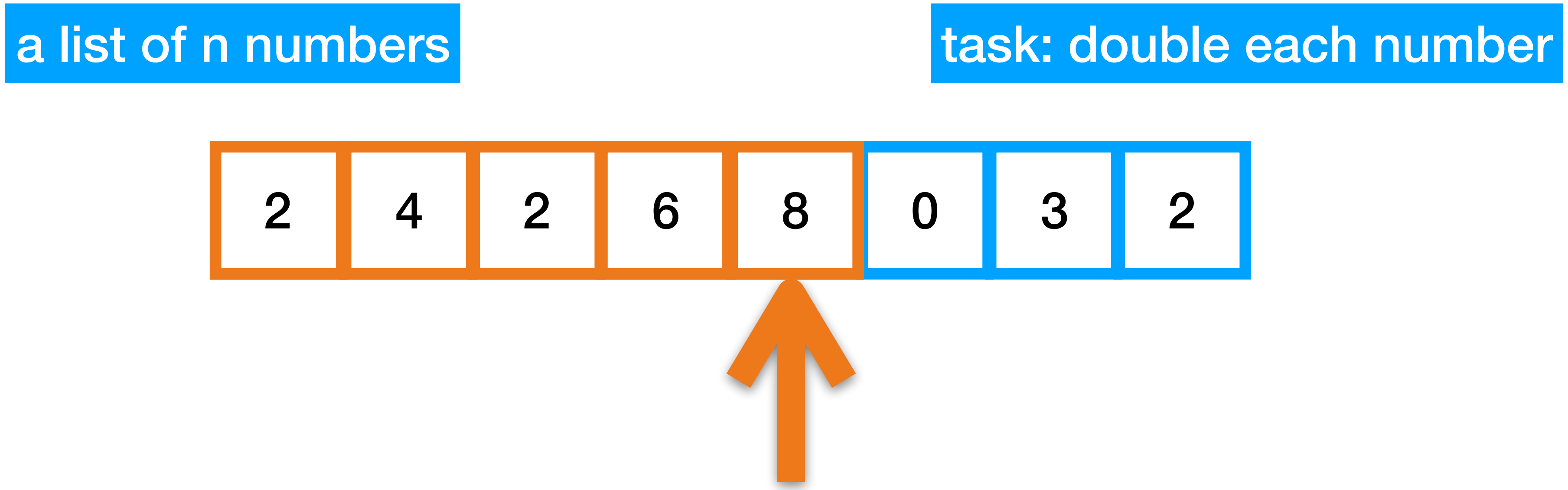

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2	4	2	6	8	0	3	2

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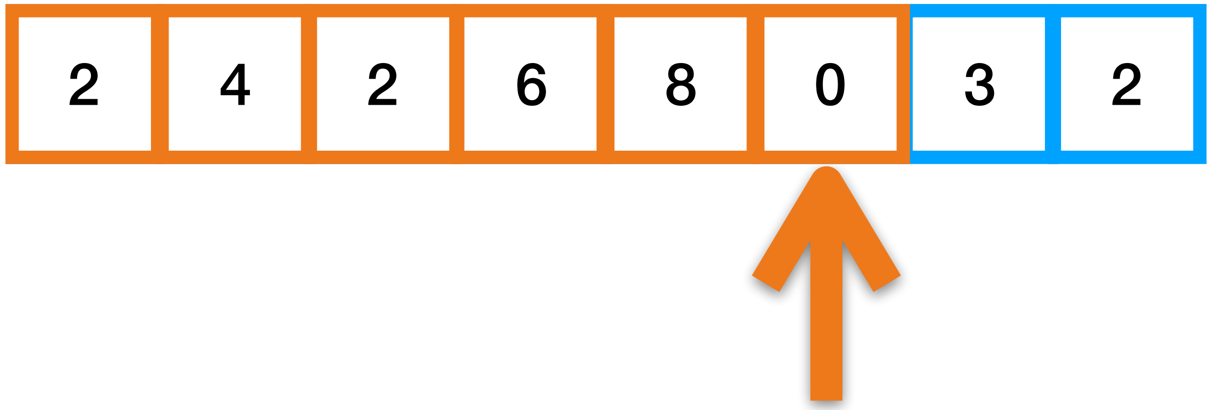

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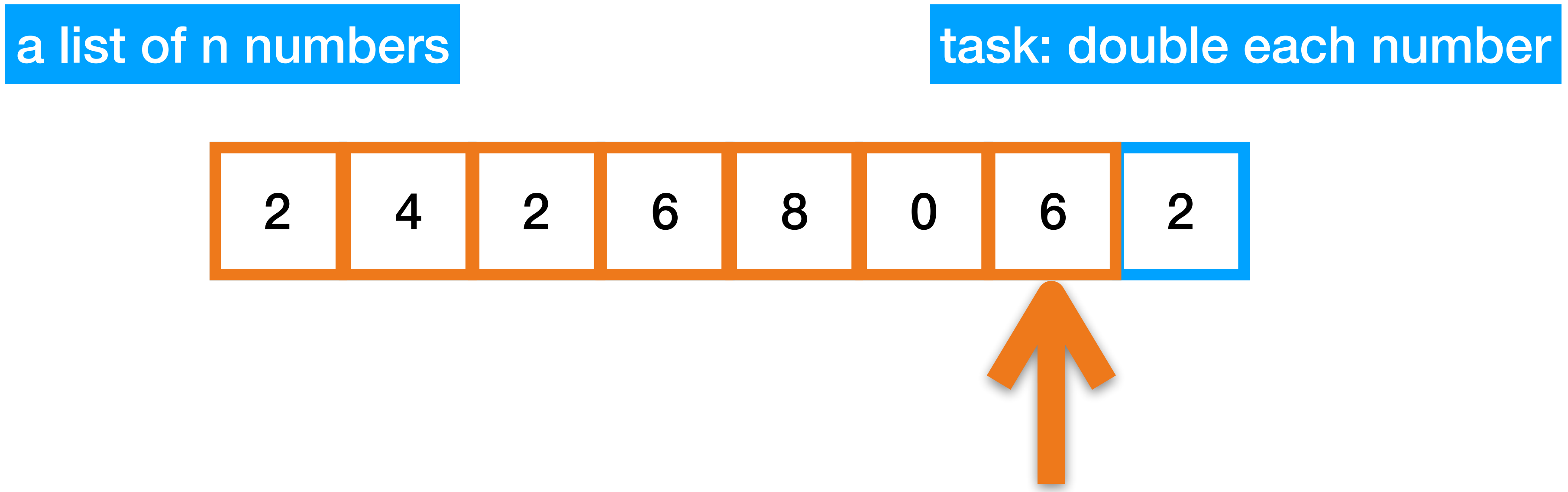

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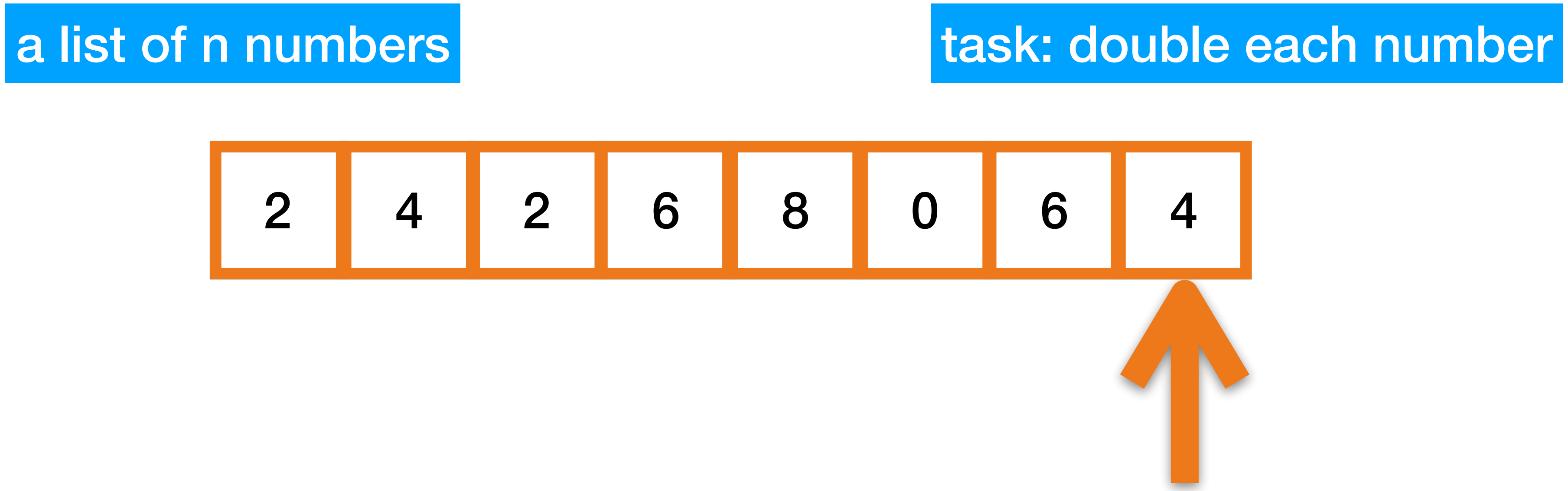

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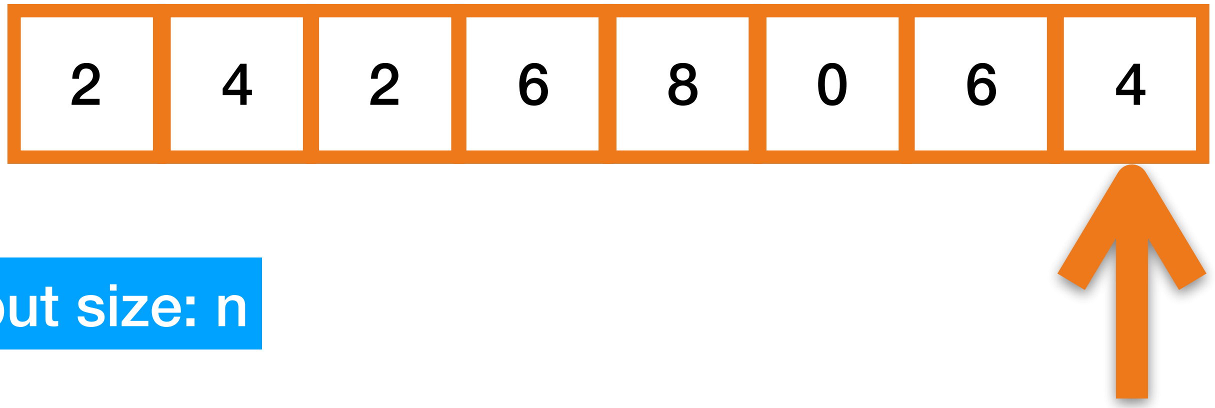

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input size: n

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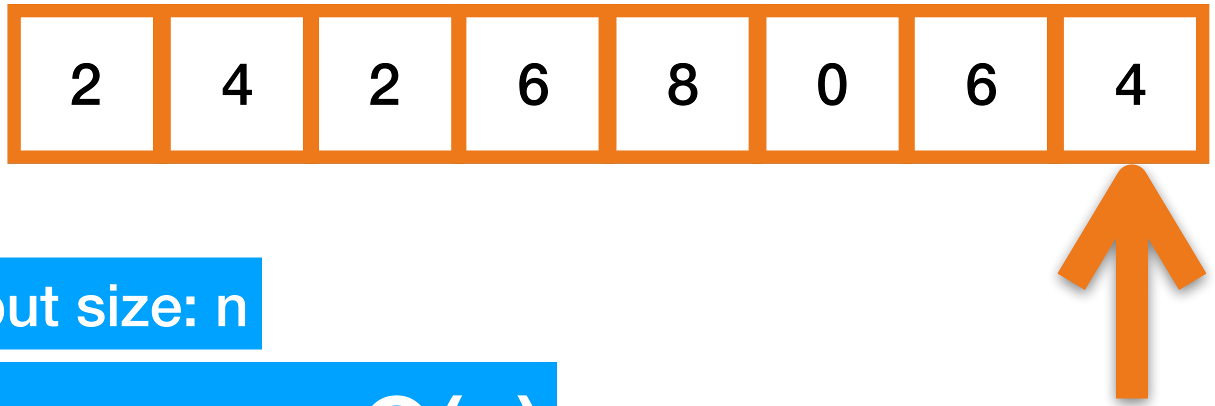

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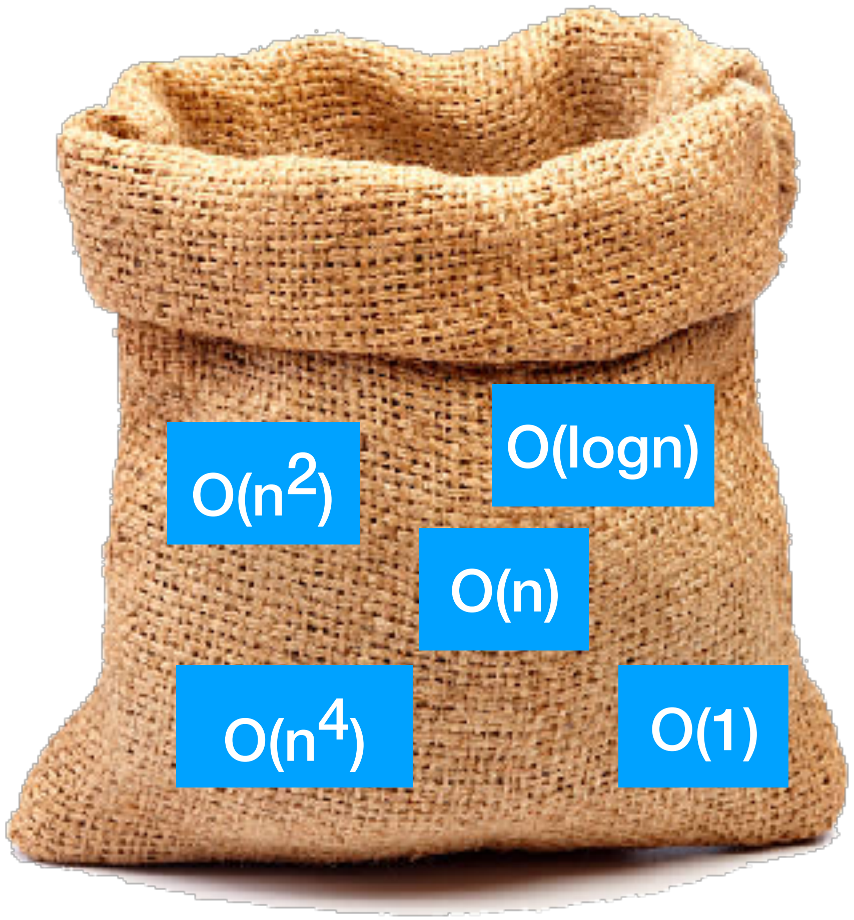
input size: n

algorithm time complexity: O(n)

Efficient algorithms

Efficient algorithms

efficient



$O(n^2)$

$O(\log n)$

$O(n)$

$O(n^4)$

$O(1)$

inefficient

$O(2^n)$

$O(n!)$

$O(n^n)$

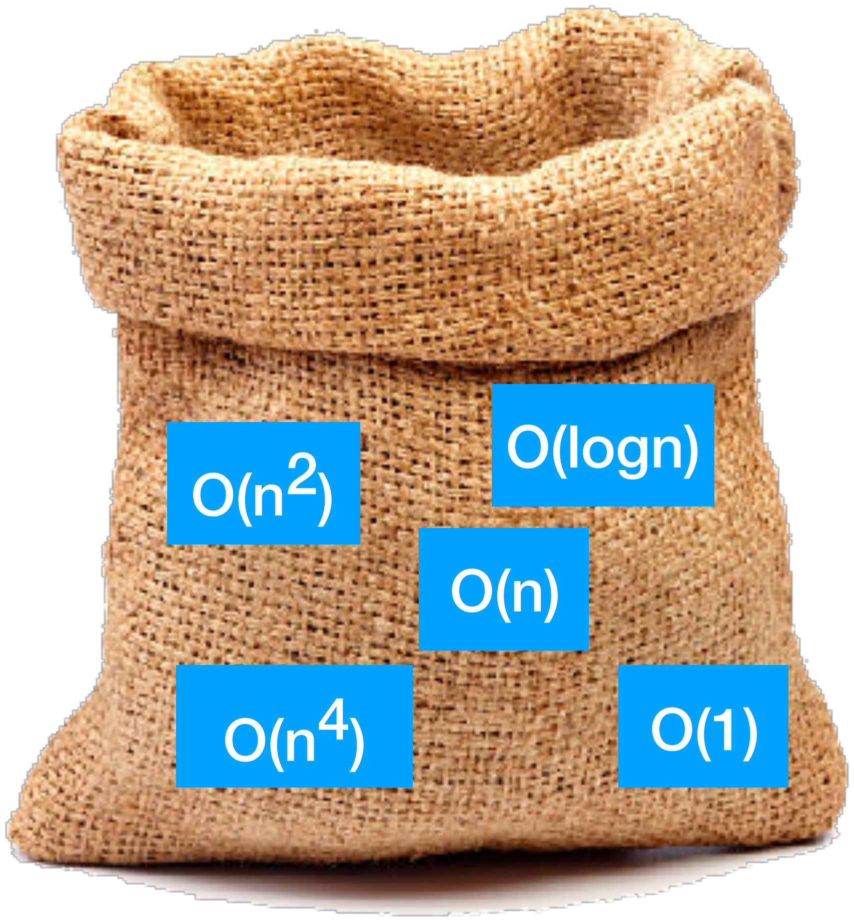
Efficient algorithms

efficient

polynomial time

inefficient

exponential time



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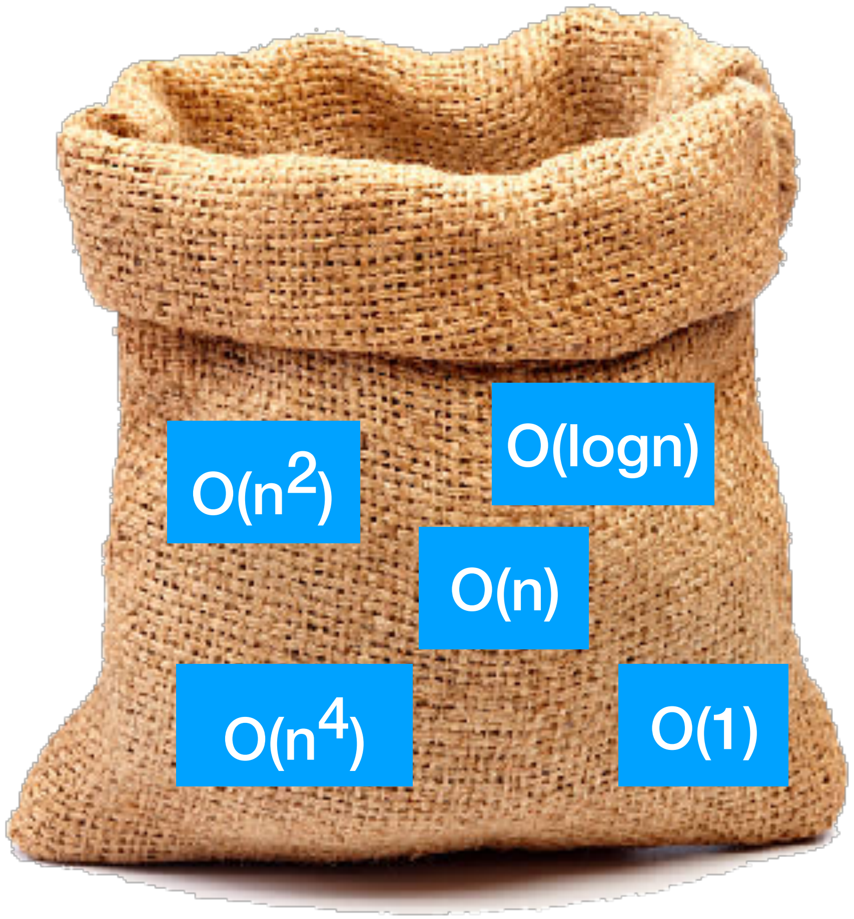
$O(n^n)$

Efficient algorithms

efficient

polynomial time

fast in general



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exponential time

slow in general

$O(2^n)$

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Efficient algorithms

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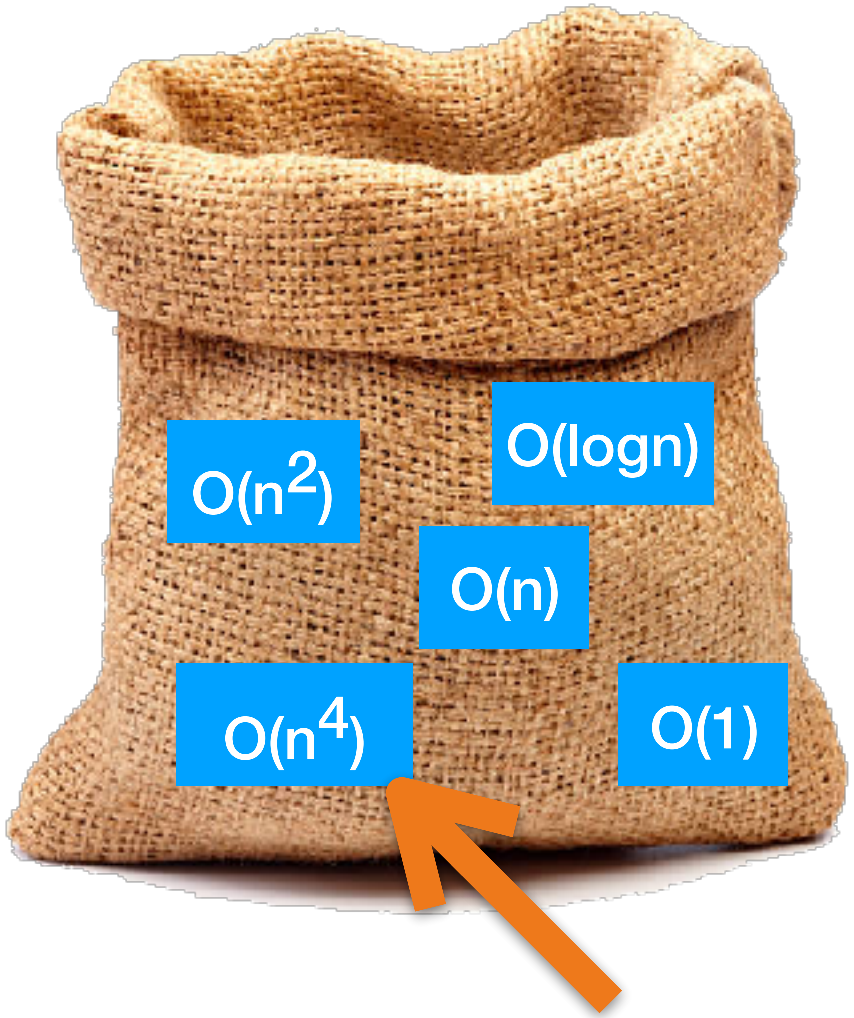

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NP hardness

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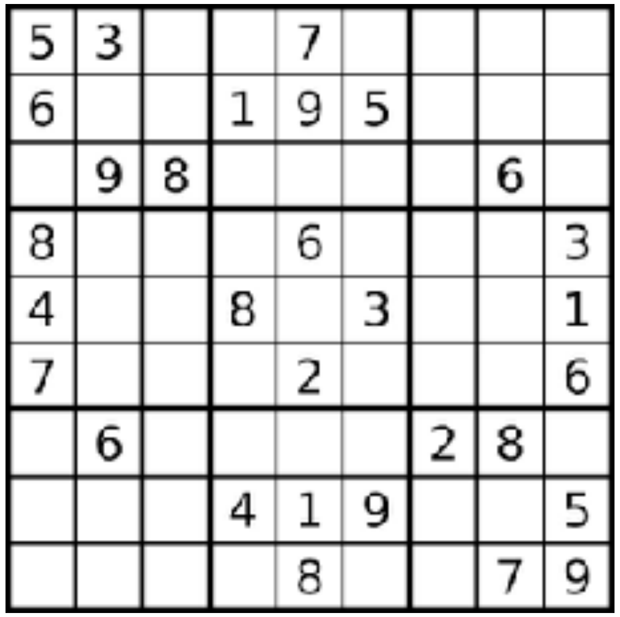
- If a problem is **NP hard**, then there is no known efficient algorithm that can solve it



5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

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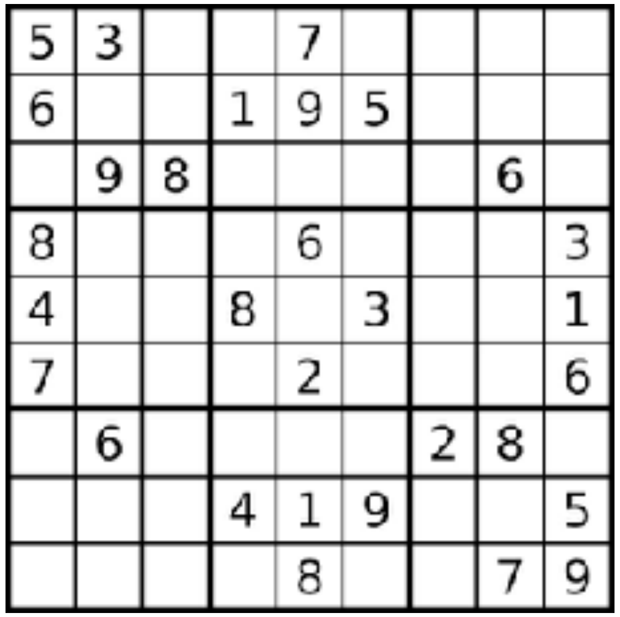
- Is it possible for an efficient algorithm to solve an **NP hard** problem?



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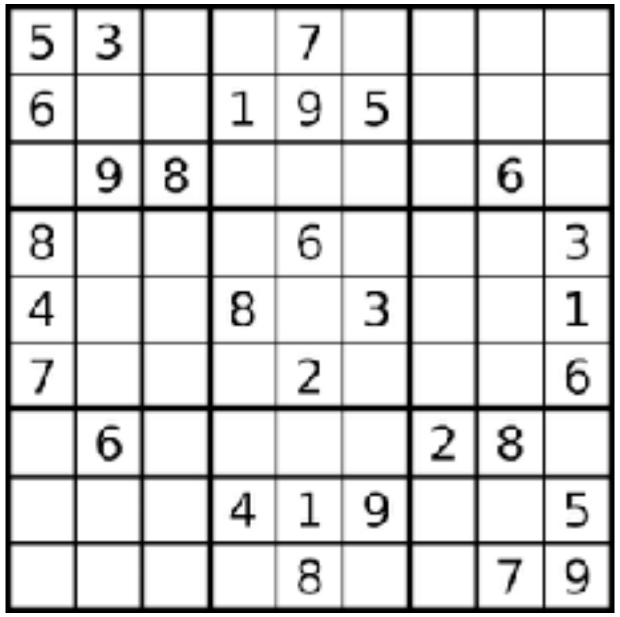

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This is one of the biggest open questions in computing science

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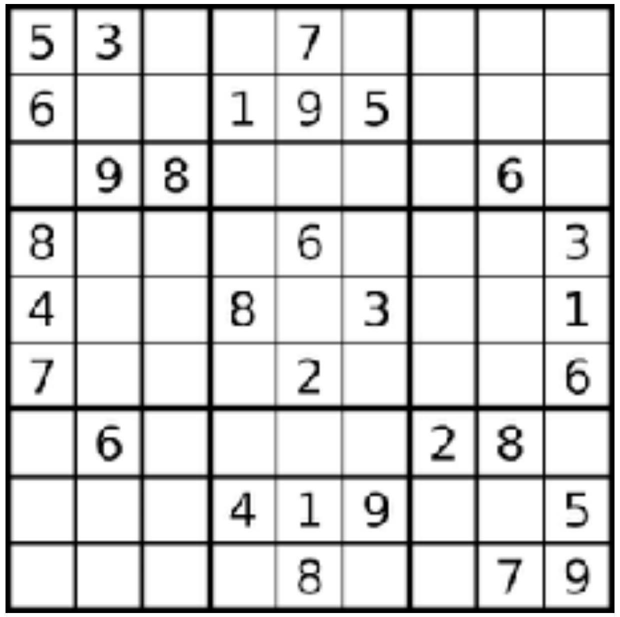

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				8			7	9

This is one of the biggest open questions in computing science

http://www.claymath.org/millennium-problems

$1,000,000 prize

Matching Problems

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- Assign one group of things to another group of things

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- Based on preferences

Student-project allocation problem

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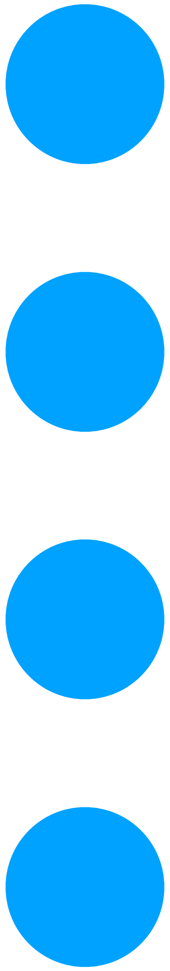
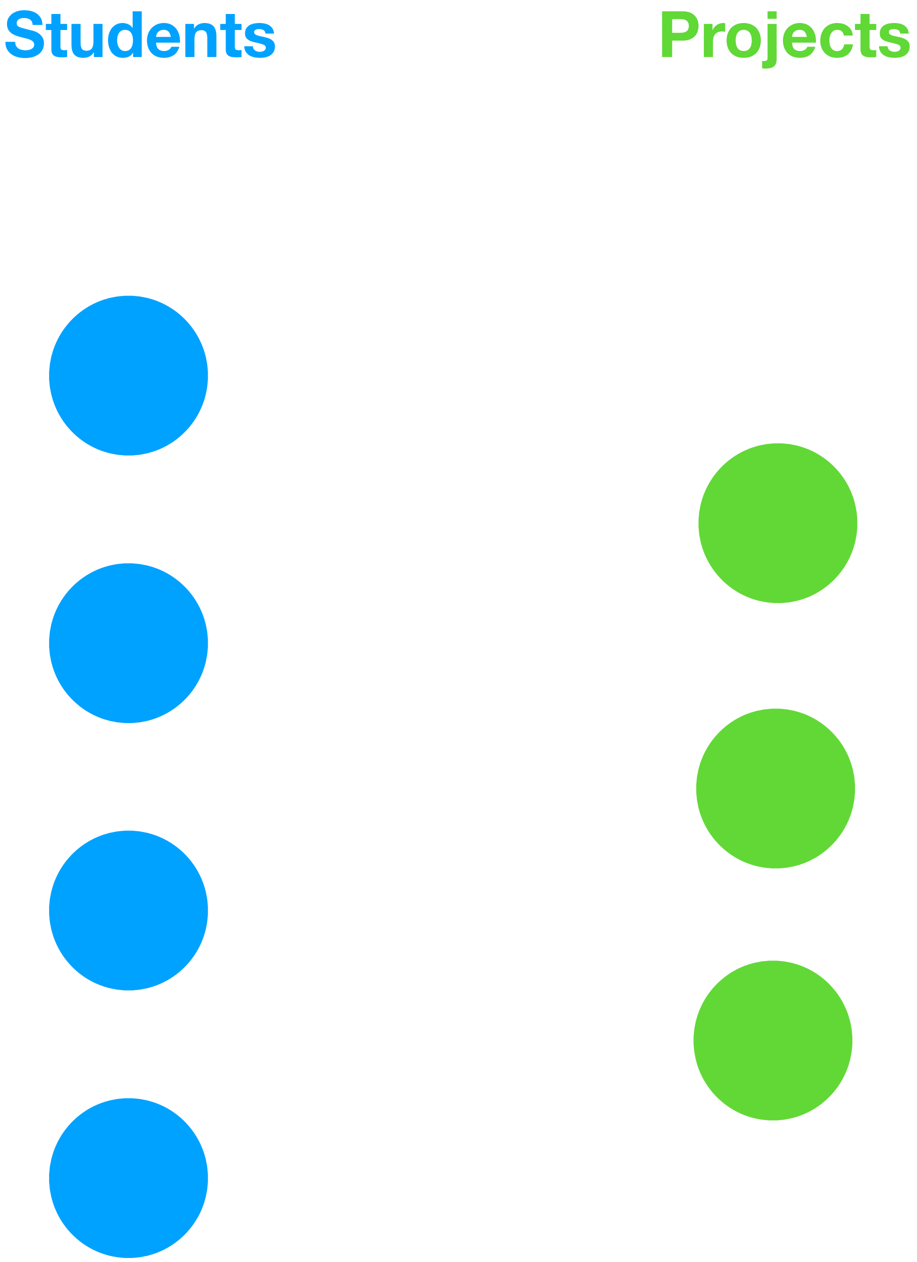


Students

Student-project allocation problem



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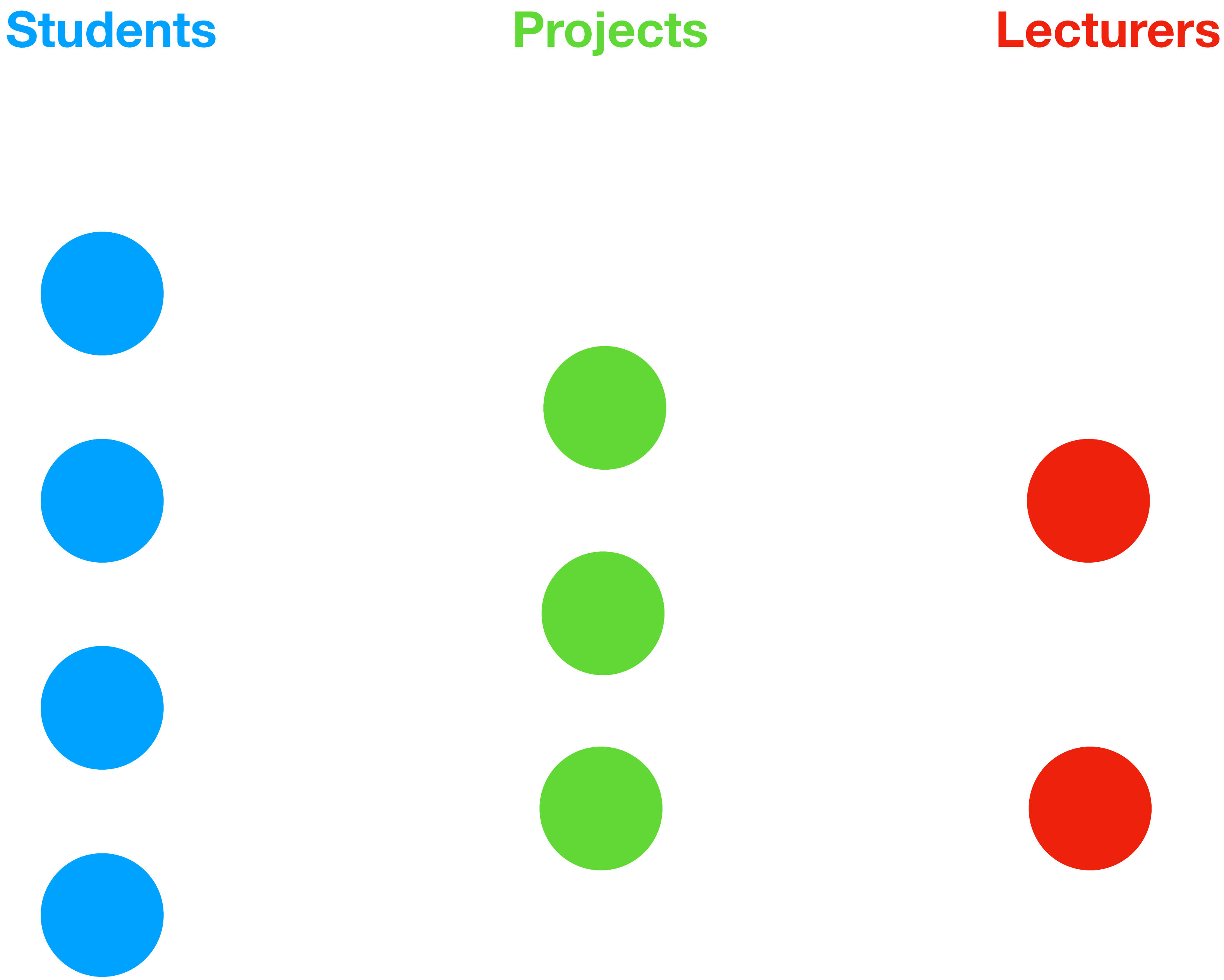
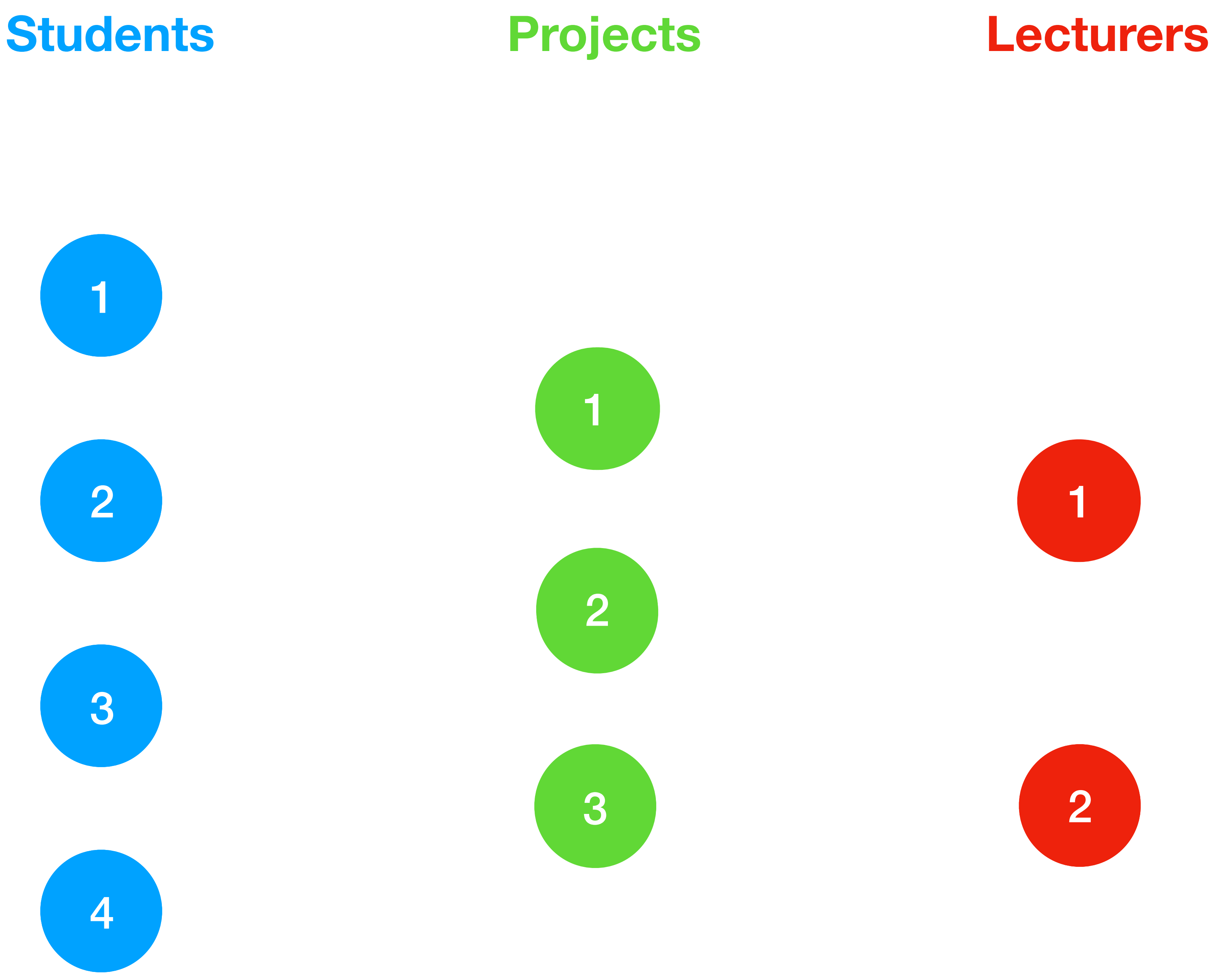


Students

Projects

Lecturers

Student-project allocation problem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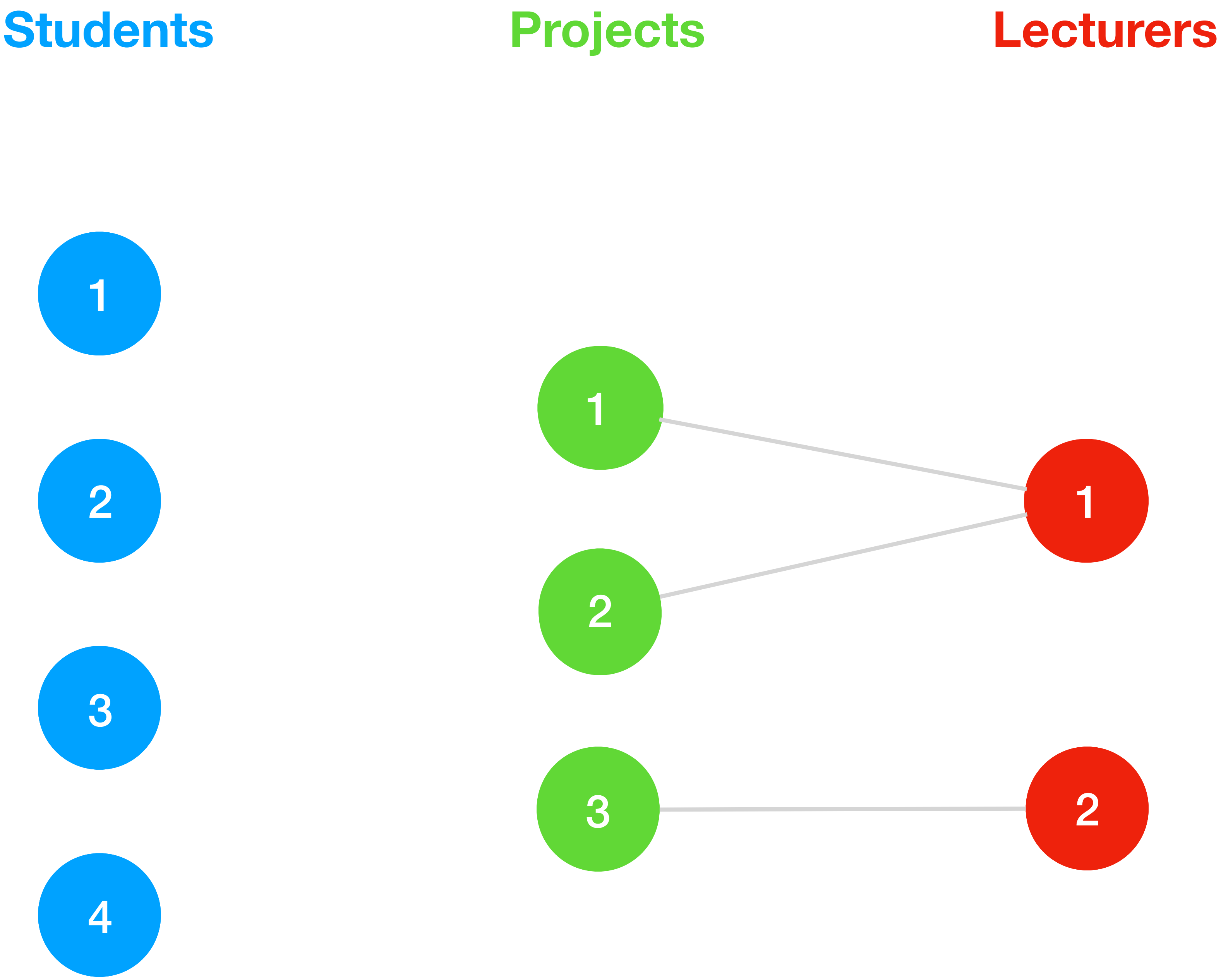

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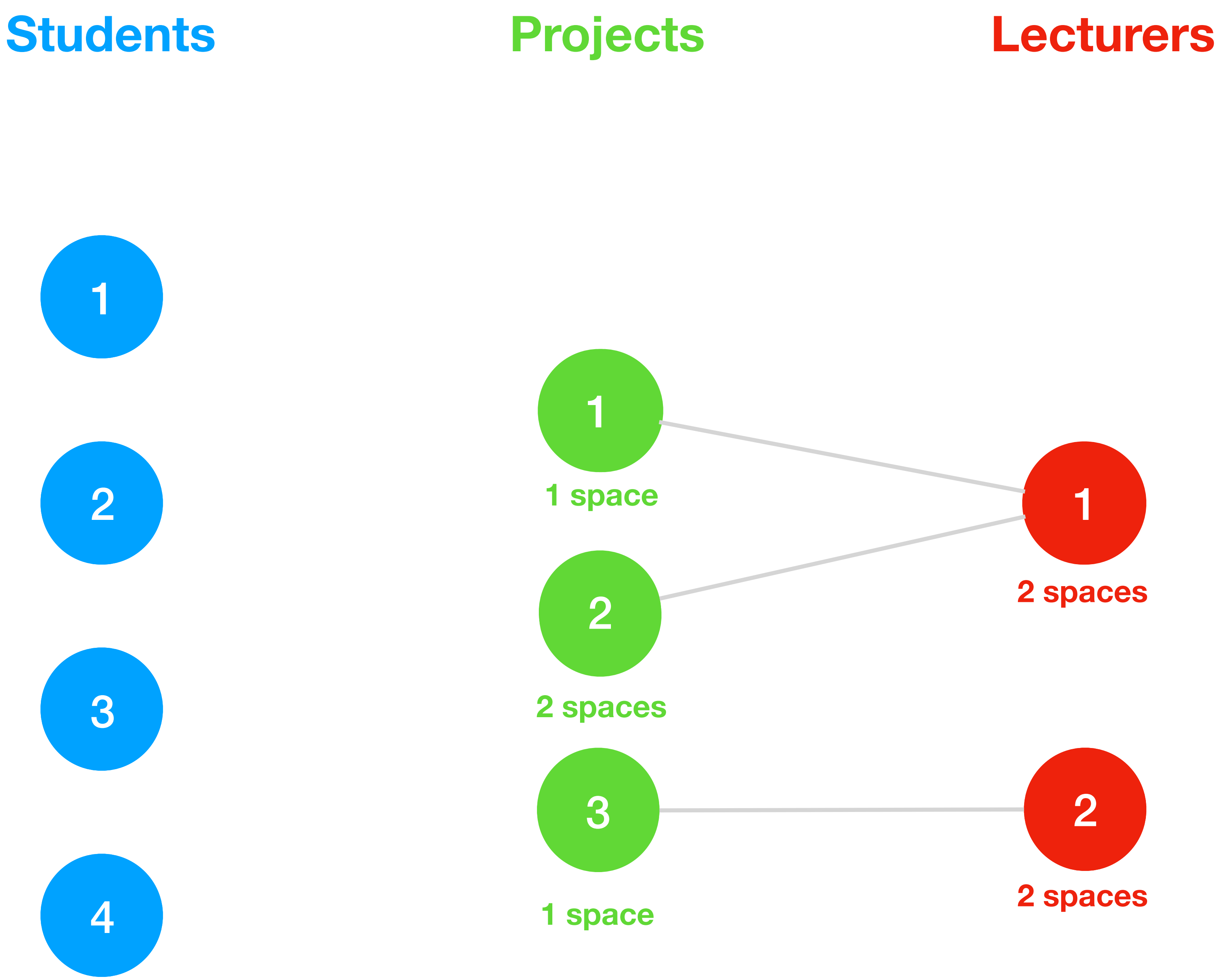

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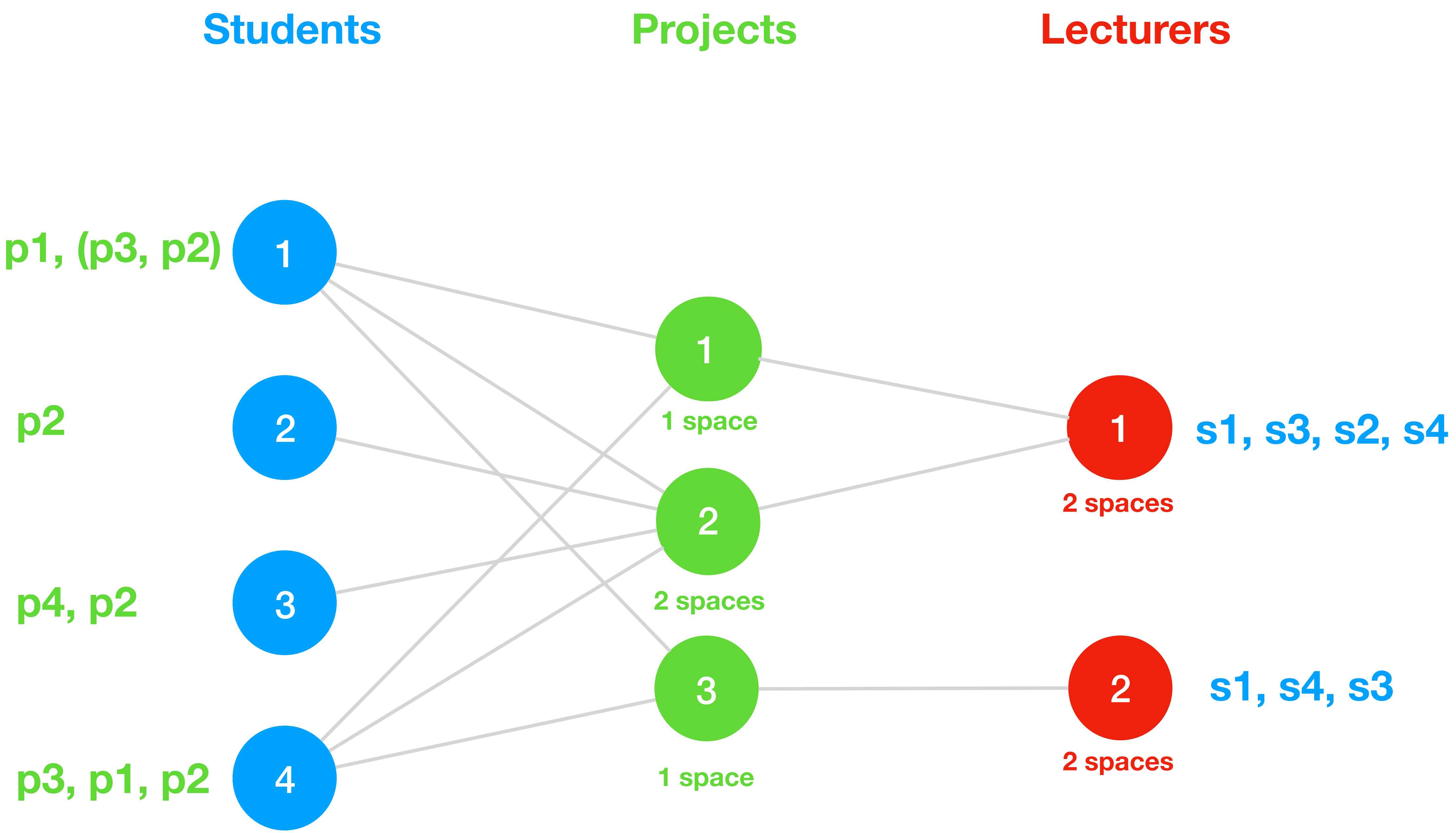

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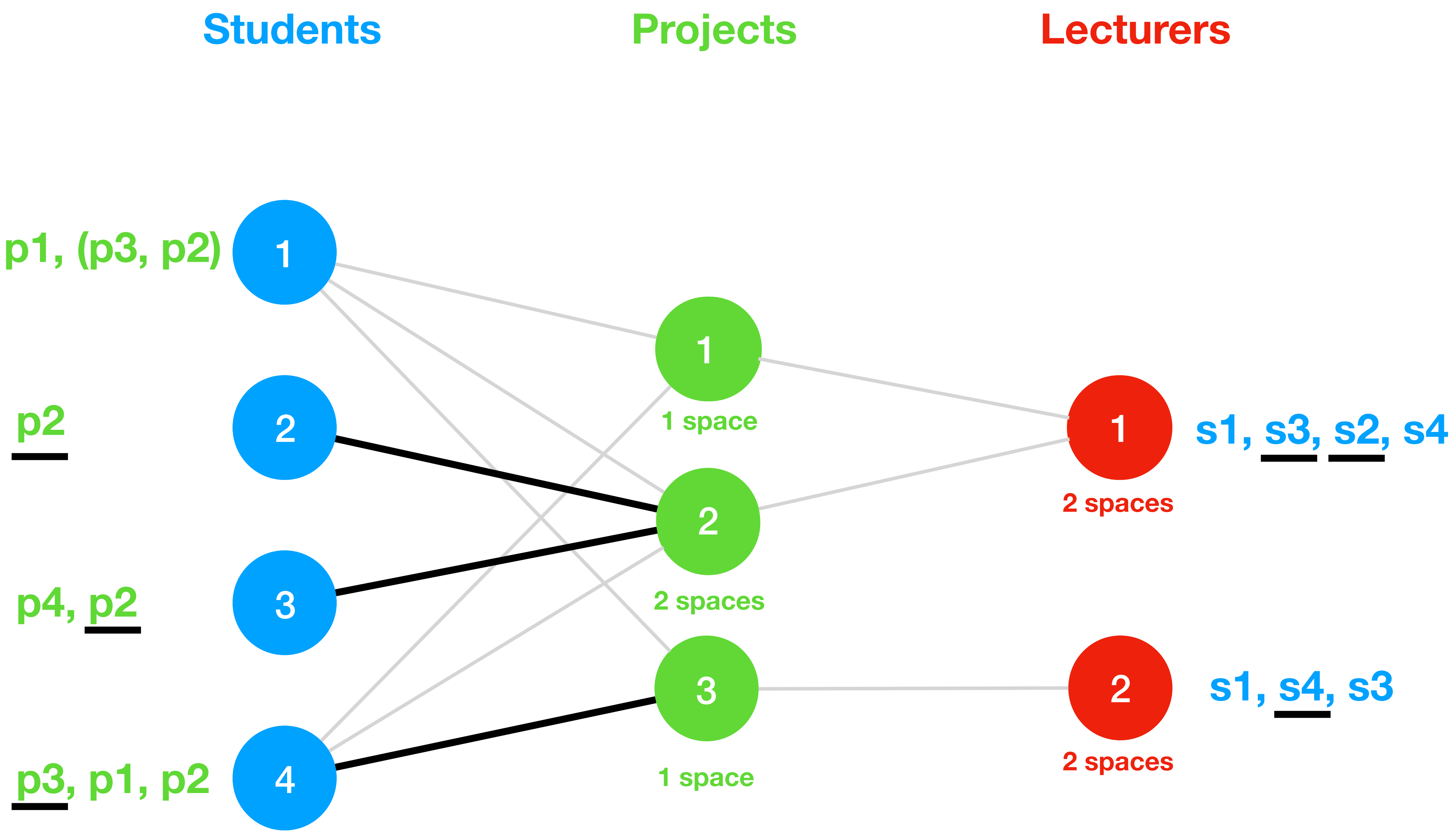

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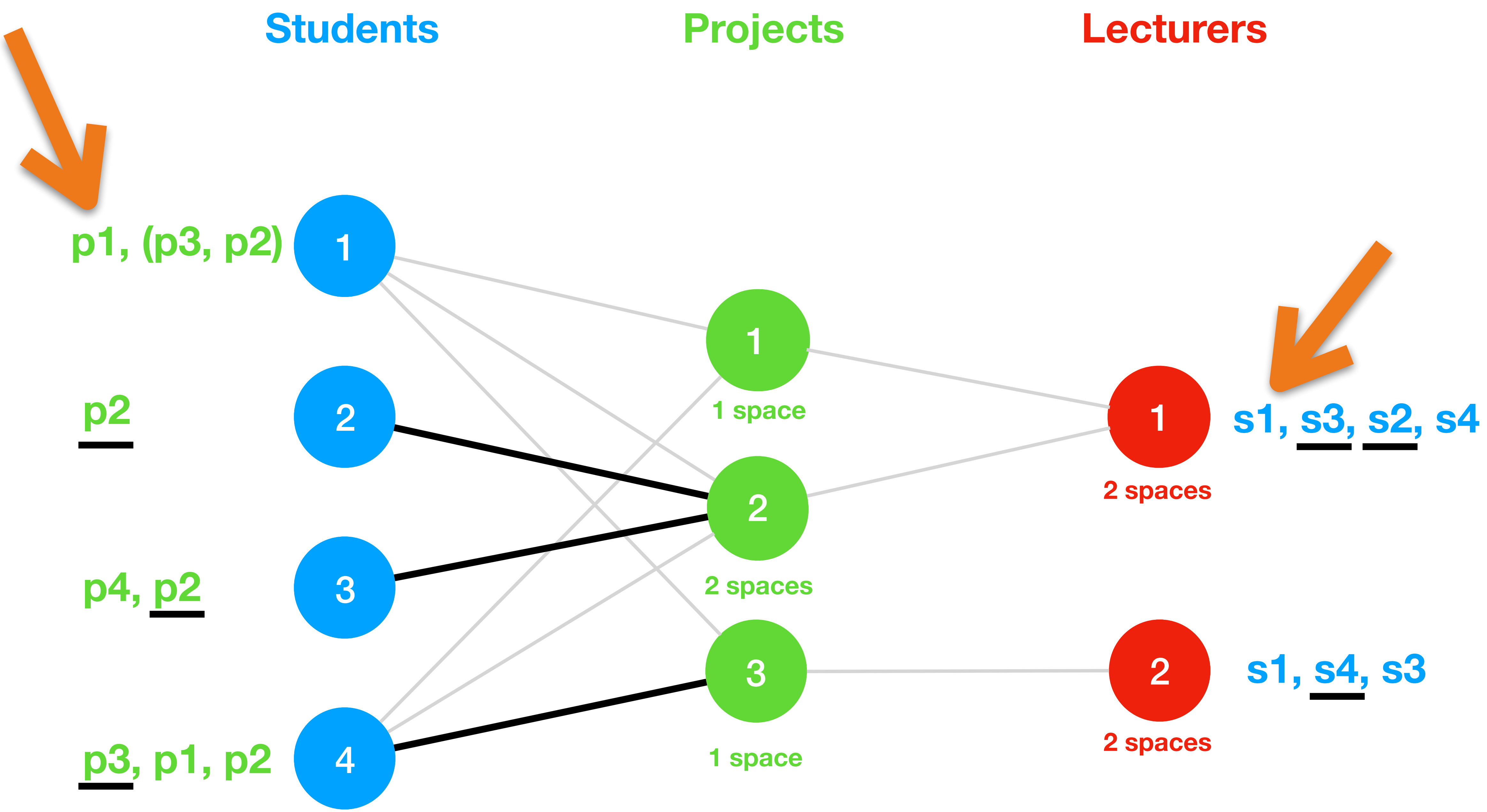
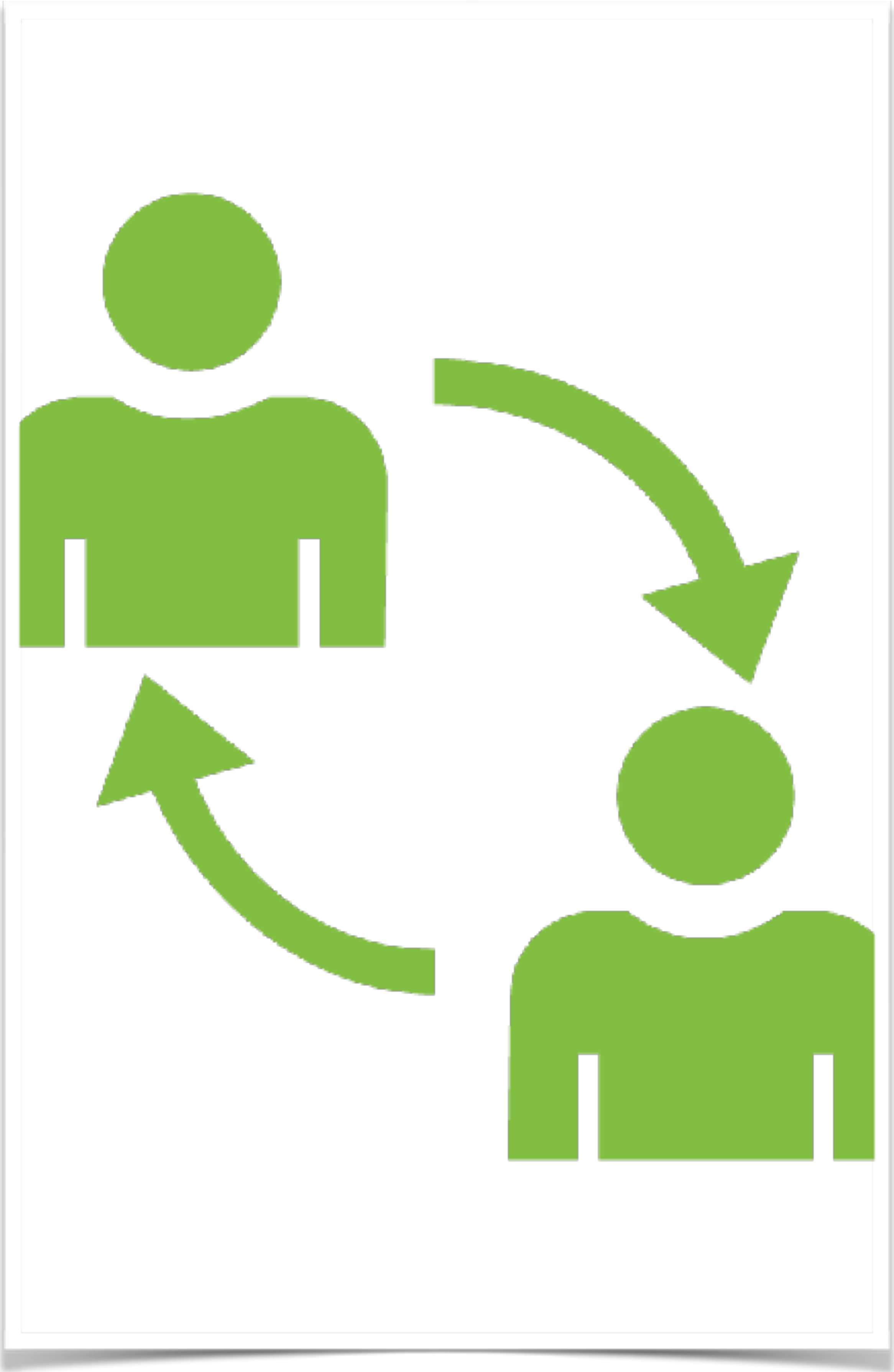


Stable matching



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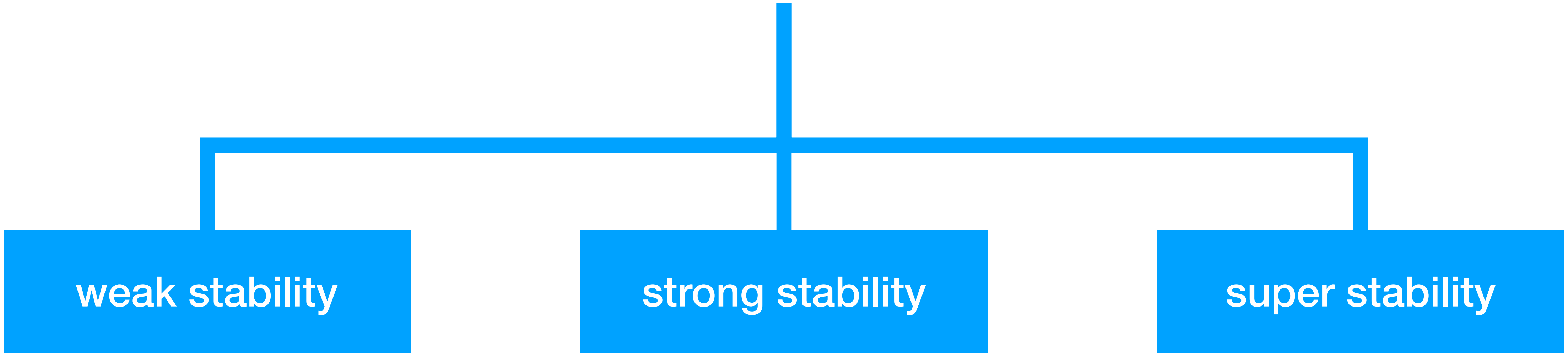
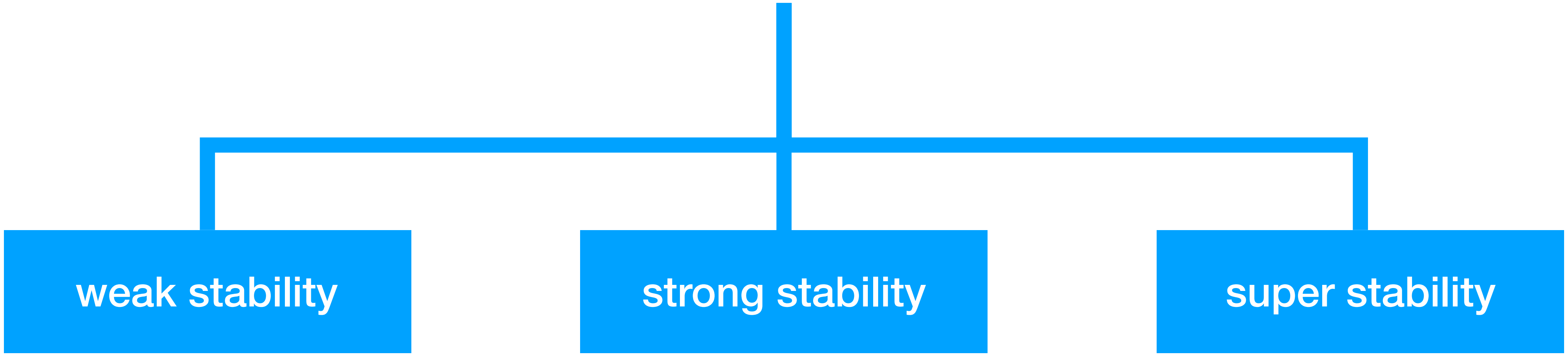

weak stability

strong stability

super stability

Stable matching



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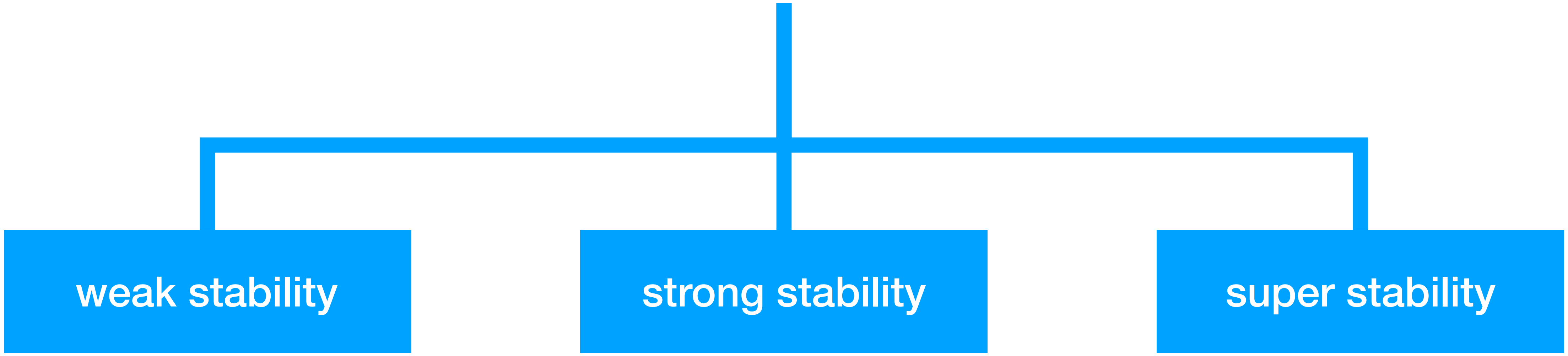
- A **stable matching** is a matching with no blocking pairs

Stable matching



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No Blocking pairs: both agents benefit

No Blocking Pairs: one agent benefits, the other is indifferent

No Blocking Pairs: neither agent is disadvantaged

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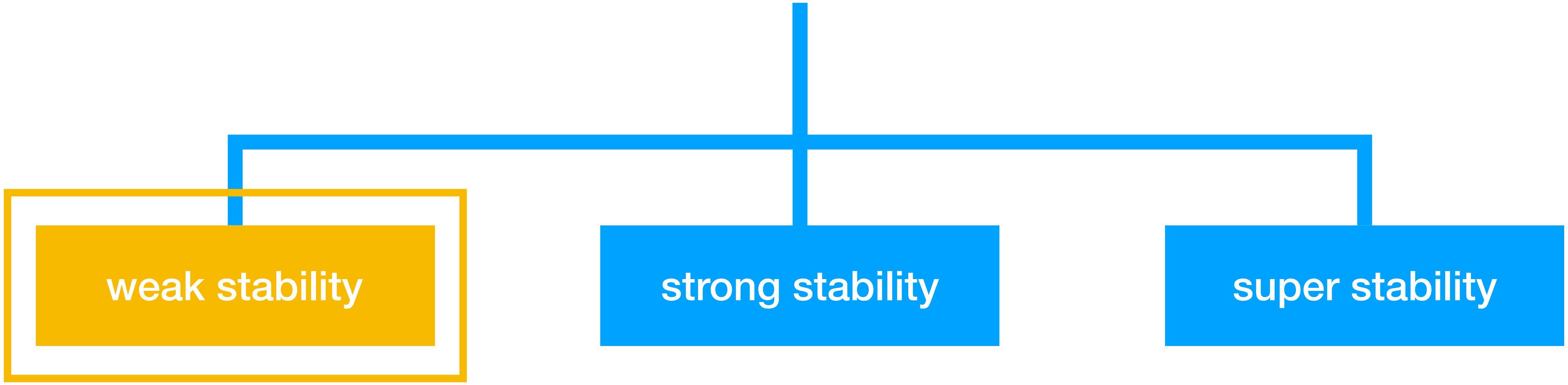

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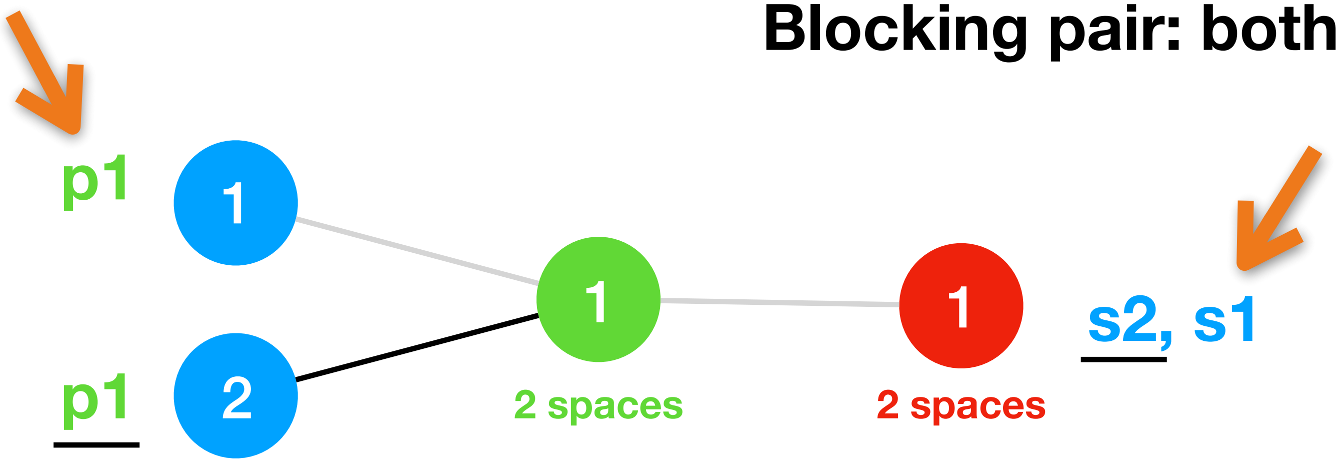
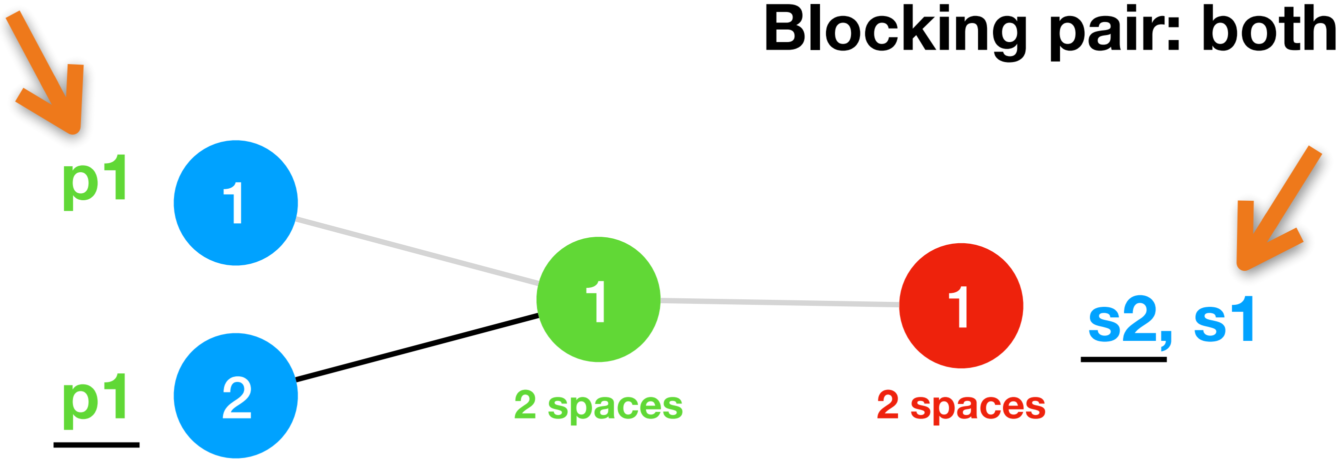


project and lecturer undersubscribed

weak stability

Blocking pair: both agents benefit



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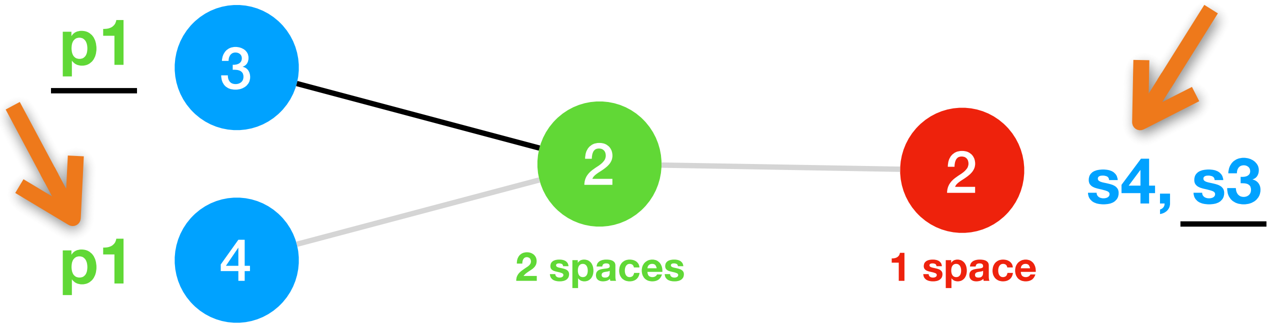

project undersubscribed, lecturer full

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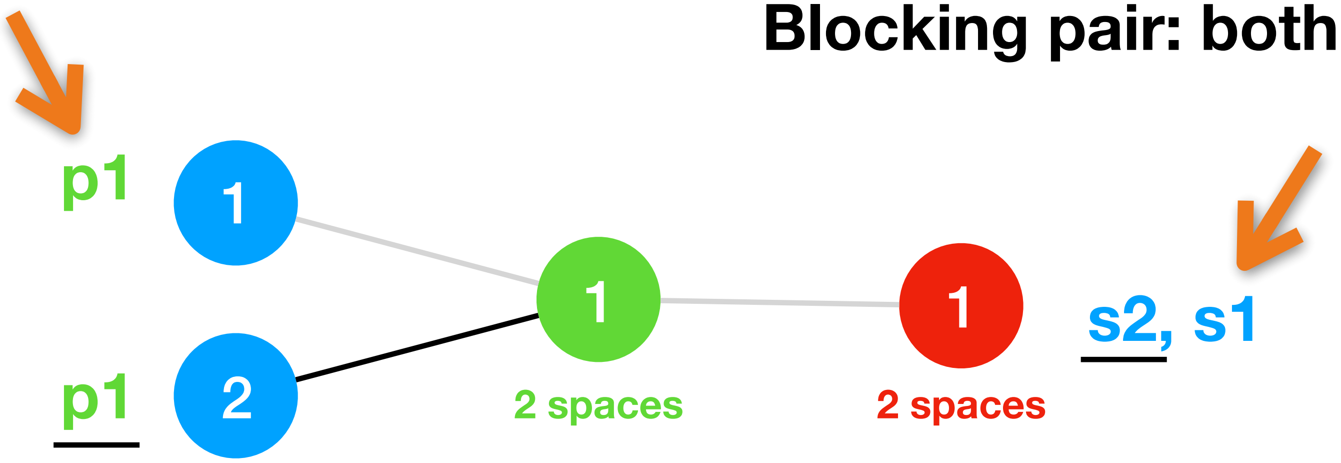

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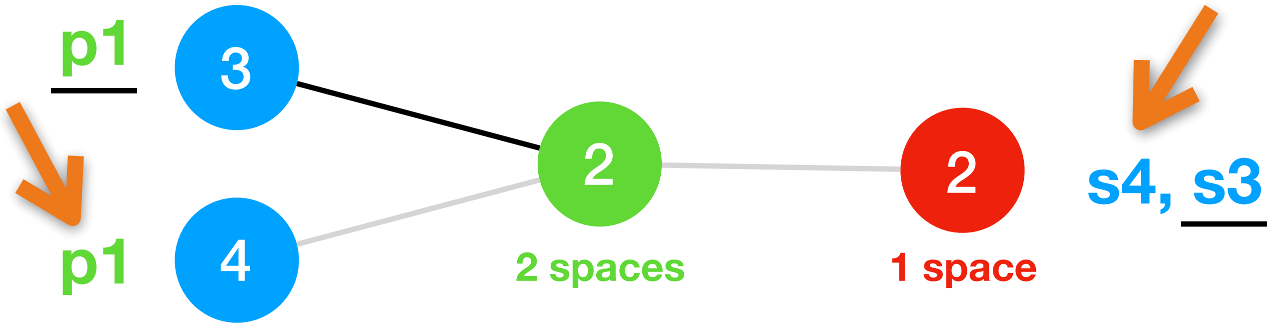

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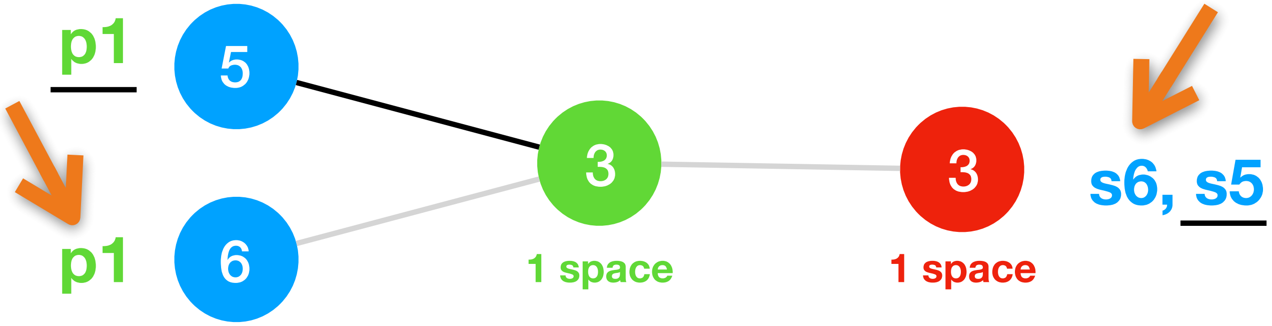
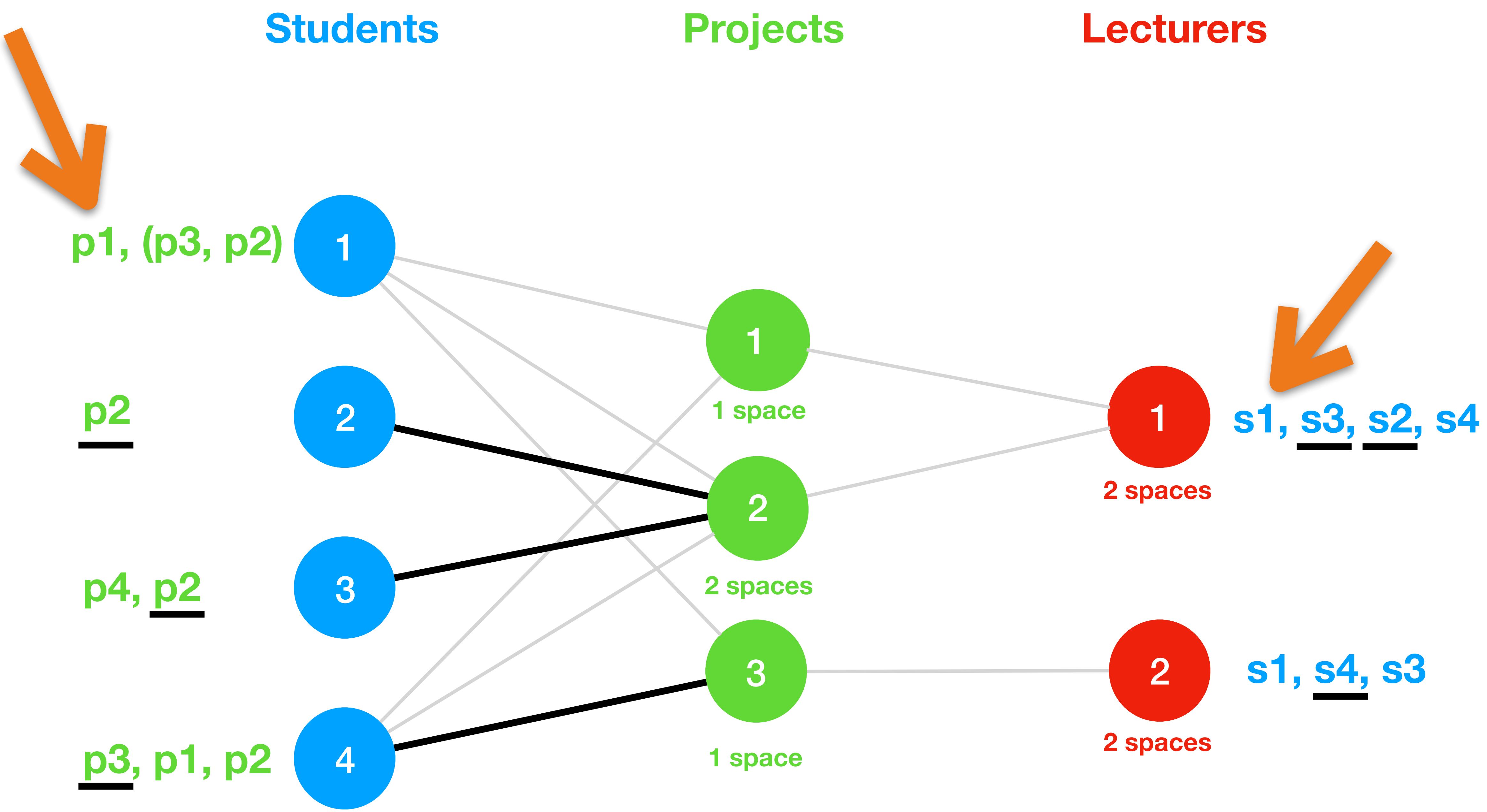


project full

Student-project allocation problem



Stable matchings

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Two Algorithms for the Student Project Allocation Problem;
Journal of Discrete Algorithms; 2007; Abraham, Irving, Manlove

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- When there are **ties in preference lists** we can find a stable matching using an **efficient algorithm** - but stable matchings are **different sizes**

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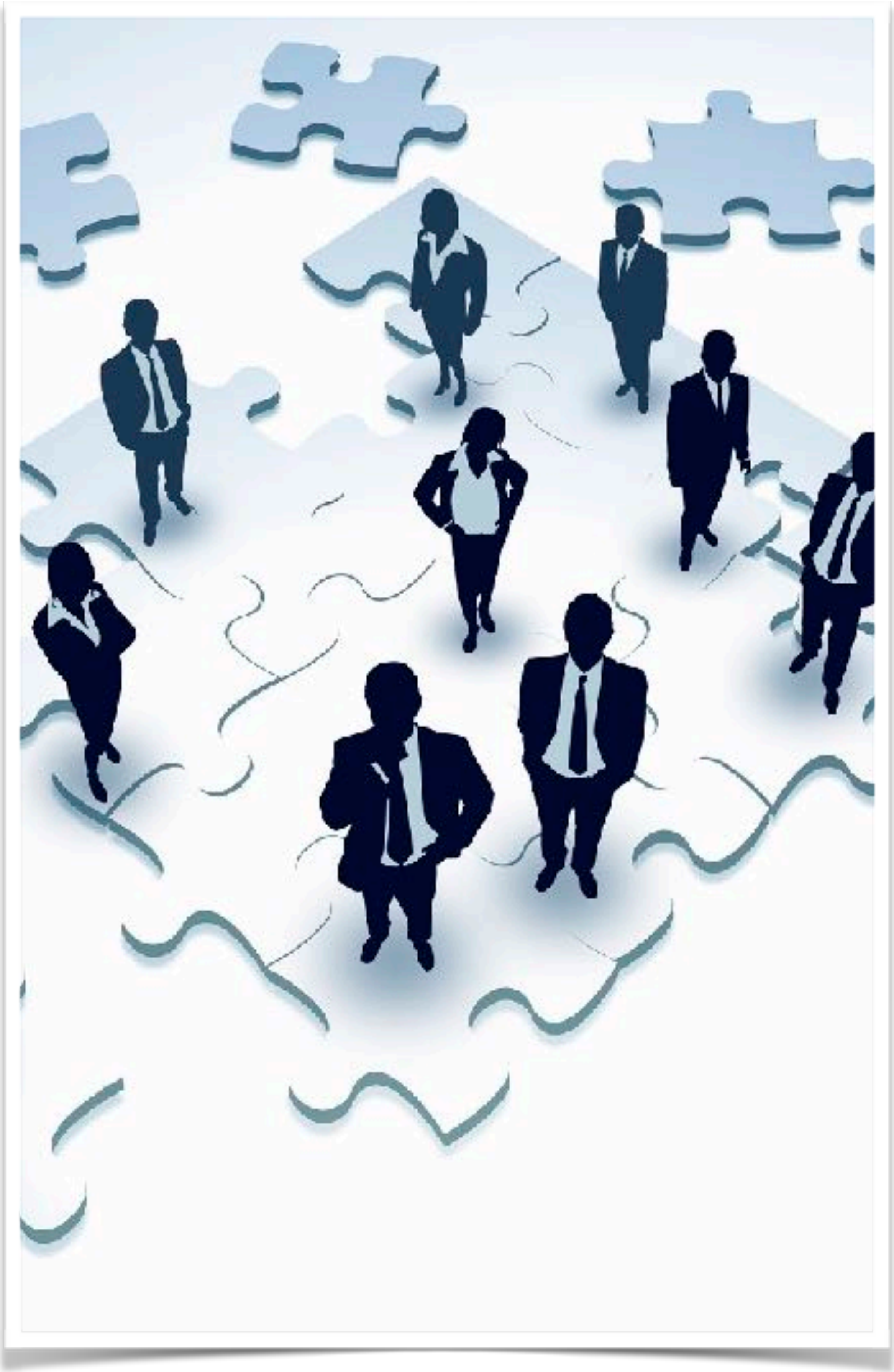
Gives us new questions

Maximum sized stable matching



Maximum sized stable matchings

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- Find an efficient algorithm for an NP-hard problem **tricky** **$1,000,000 prize**

Maximum sized stable matchings

Finding a maximum sized stable matching is **NP-hard**. What can we do?

- Build an exhaustive search algorithm inefficient
- Find an efficient algorithm for an NP-hard problem tricky $1,000,000 prize
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- Integer programming (uses inefficient algorithms)
- Approximation algorithms (poly time)

Integer Programming

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- An Integer programming model has been built to find a **maximum stable matching**

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positive: **efficient** algorithm

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    - Can I just convert my problem and use this? No!

- Had to create a new 3/2 approximation algorithm
    - Lecturers added a lot of complications
    - Proved that this algorithm is efficient (polynomial-time) and correct (results in a stable matching at least 2/3 the size of a **maximum stable matching**)

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- if either **p** or **l** are full then we need to check whether (**s**,**p**) should replace an *existing* pair in the matching
- if there is no chance for **s** to assign to **p** then **s** will remove **p** from their preference list (and will now apply to their next favourite)
- Students iterate twice through their preference list

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- Much faster than using the integer program
- So is it worth using?

Future Work: coalitions

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- group of several students and lecturers

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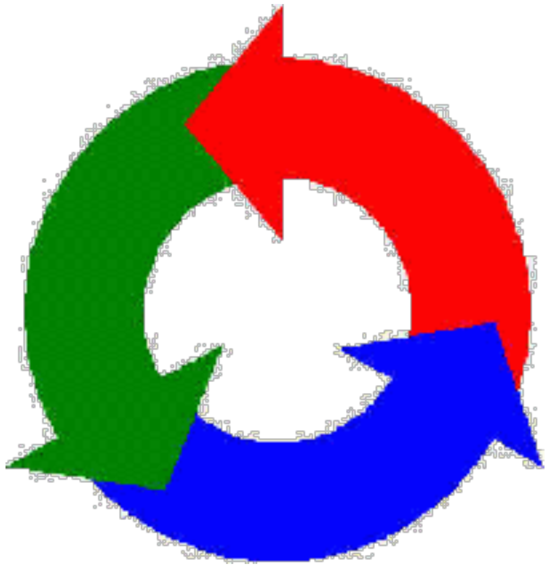
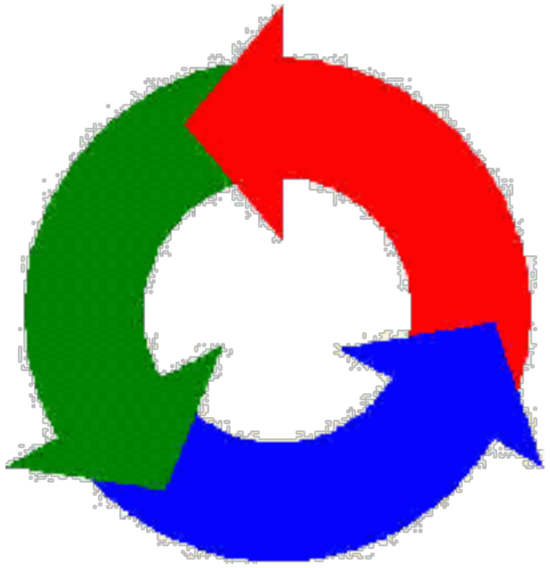
- permute their assignments



Future Work: coalitions

- group of several students and lecturers
- permute their assignments
- some or all get a better outcome



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    - Approximation algorithm

University of Glasgow

VIA VERITAS VITA

Thank you

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