# Regret-equality in Stable Marriage

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# Outline

- Matching problems
- Fairness
- Finding fair stable matchings
- Experiments
- Future work

# Matching Problems


- Assign one set of entities to another set of entities
- Based on preferences and capacities

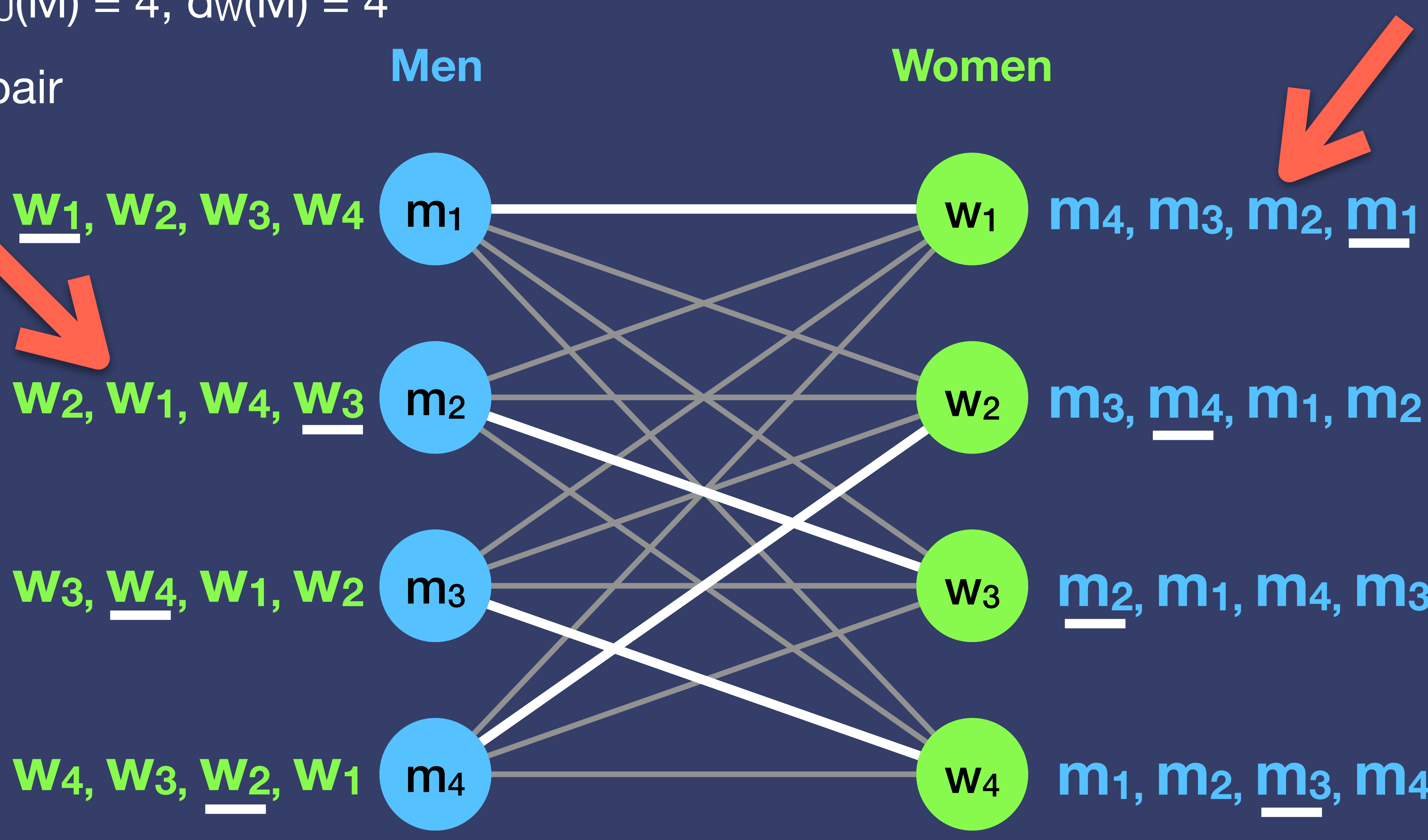
# Stable Marriage

Rank

Cost: $c_U(M) = 10$, $c_W(M) = 10$

Degree: $d_U(M) = 4$, $d_W(M) = 4$

Blocking pair




A **stable matching** is a matching with no blocking pairs



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# Stable Marriage

- A **stable matching** is a matching with no blocking pairs

- Many stable matchings per instance

- We can find a stable matching in linear time using the man-oriented or woman-oriented Gale-Shapley Algorithm. O(m) time where m is total length of preference lists

- Man-oriented Gale-Shapley Algorithm: finds a man-optimal (woman-pessimal) stable matching (and vice versa)

# Fairness

- Want to find a stable matching that provides some kind of equality between men and women

- Several different fairness measures

# Fairness measures

Among all stable matchings, find the stable matching that…

| | Cost: $c_U(M)$, $c_W(M)$ | Degree: $d_U(M)$, $d_W(M)$ |
|---|---|---|
| Minimises the maximum | **balanced score**<br>Balanced stable matching NP-hard | **degree**<br>Minimum-regret stable matching Poly |
| Minimises the difference | **sex-equal score**<br>Sex-equal stable matching NP-hard | **regret-equal score**<br>* Regret-equal stable matching ? |
| Minimises the sum | **egalitarian cost**<br>Egalitarian stable matching Poly | **regret sum score**<br>* Min-regret sum stable matching ? |

# Fairness measures (degree based)

10 stable matchings for this instance

$m_1$: $w_1, w_2, \underline{w_3}, w_4$ | $w_1$: $m_4, m_3, \underline{m_2}, m_1$
$m_2$: $w_2, \underline{w_1}, w_4, w_3$ | $w_2$: $m_3, \underline{m_4}, m_2, m_1$
$m_3$: $w_3, \underline{w_4}, w_1, w_2$ | $w_3$: $m_2, \underline{m_1}, m_4, m_3$
$m_4$: $w_4, w_3, \underline{w_2}, w_1$ | $w_4$: $m_1, m_2, \underline{m_3}, m_4$

**Min-regret & Regret-equal**

Degree: 3

Regret-equality score: 0

Min-regret sum score: 6

$m_1$: $w_1, w_2, \underline{w_3}, w_4$ | $w_1$: $m_4, \underline{m_3}, m_2, m_1$
$m_2$: $w_2, w_1, \underline{w_4}, w_3$ | $w_2$: $m_3, \underline{m_4}, m_2, m_1$
$m_3$: $w_3, w_4, \underline{w_1}, w_2$ | $w_3$: $m_2, \underline{m_1}, m_4, m_3$
$m_4$: $w_4, w_3, \underline{w_2}, w_1$ | $w_4$: $m_1, \underline{m_2}, m_3, m_4$

**Min-regret & Min-regret sum**

Degree: 3

Regret-equality score: 1

Min-regret sum score: 5

$m_1$: $\underline{w_1}, w_2, w_3, w_4$ | $w_1$: $m_4, m_3, m_2, \underline{m_1}$
$m_2$: $\underline{w_2}, w_1, w_4, w_3$ | $w_2$: $m_3, m_4, \underline{m_2}, m_1$
$m_3$: $\underline{w_3}, w_4, w_1, w_2$ | $w_3$: $m_2, m_1, m_4, \underline{m_3}$
$m_4$: $\underline{w_4}, w_3, w_2, w_1$ | $w_4$: $m_1, m_2, m_3, \underline{m_4}$

**Min-regret sum**

Degree: 4

Regret-equality score: 3

Min-regret sum score: 5

Over all stable matchings:
Minimum degree = 3
Minimum regret-equality score = 0
Minimum regret sum score = 5

# Finding a Regret-Equal Stable Matching

# Rotations

- Rotation - series of man-woman pairs that take us from one stable matching to another when permuted

$$R_1 \quad \begin{array}{cc} m_1 & m_4 \\ w_2 & w_3 \end{array}$$

- Can only eliminate *exposed* rotations

$$R_2 \quad \begin{array}{cc} m_1 & m_2 \\ w_1 & w_2 \end{array}$$

$$M_1 \quad \begin{array}{cccc} m_1 & m_2 & m_3 & m_4 \\ w_2 & w_1 & w_4 & w_3 \end{array}$$

$$\Big\downarrow R_1$$

$$M_2 \quad \begin{array}{cccc} m_1 & m_2 & m_3 & m_4 \\ w_3 & w_1 & w_4 & w_2 \end{array}$$

- $O(n^2)$ algorithm to find all rotations

- Rotations form a structure to allow enumeration of all stable matchings. All rotation makes some men worse off and some women better off

# Algorithm

1. Find the man-optimal stable matching $M_0$

- Each man has their best partner in any stable matching. Say $d_U(M_0) = 2$ and $d_W(M_0) = 5$ $\quad d(M_0) = (2, 5)$

- Then, a regret equal stable matching must exist within the following degrees pairs:

| | |
|---|---|
| r-e score: 3 | (2, 5) |
| r-e score: 2 | (2, 4) (3, 5) |
| r-e score: 1 | (2, 3) (3, 4) (4, 5) |
| r-e score: 0 | (2, 2) (3, 3) (4, 4) (5, 5) |
| r-e score: 1 | (2, 1) (3, 2) (4, 3) (5, 4) (6, 5) |
| r-e score: 2 | (3, 1) (4, 2) (5, 3) (6, 4) (7, 5) |

why are these the only possible degrees?

- $M_0$ has a r-e score of 3
- men can only get worse
- women can only get better

# Algorithm

2. If $d_U(M_0)$ >= $d_W(M_0)$ then exit with $M_0$
3. For each man m and for each column c:
   1. rotate m down to c (if possible)
   2. rotate women down column c who have worst rank

r-e score: 3 (2, 5)
r-e score: 2 (2, 4) (3, 5)
r-e score: 1 (2, 3) (3, 4) (4, 5)
r-e score: 0 (2, 2) (3, 3) (4, 4) (5, 5)
r-e score: 1 (2, 1) (3, 2) (4, 3) (5, 4) (6, 5)
r-e score: 2 (3, 1) (4, 2) (5, 3) (6, 4) (7, 5)

- Stop iterating women up the column when $d_U(M)$ >= $d_W(M)$
- Save the best matching as you go

# Time complexity

- Find man-optimal stable matching & all rotations **O(m)**
- For each man **O(n)**
    - For each column **$O(2 * |d_U(M_0) - d_W(M_0)|) = O(c)$**

      2 * man-optimal difference
        - Rotate man up and women down **O(m)**

★ Total O(nmc) ★


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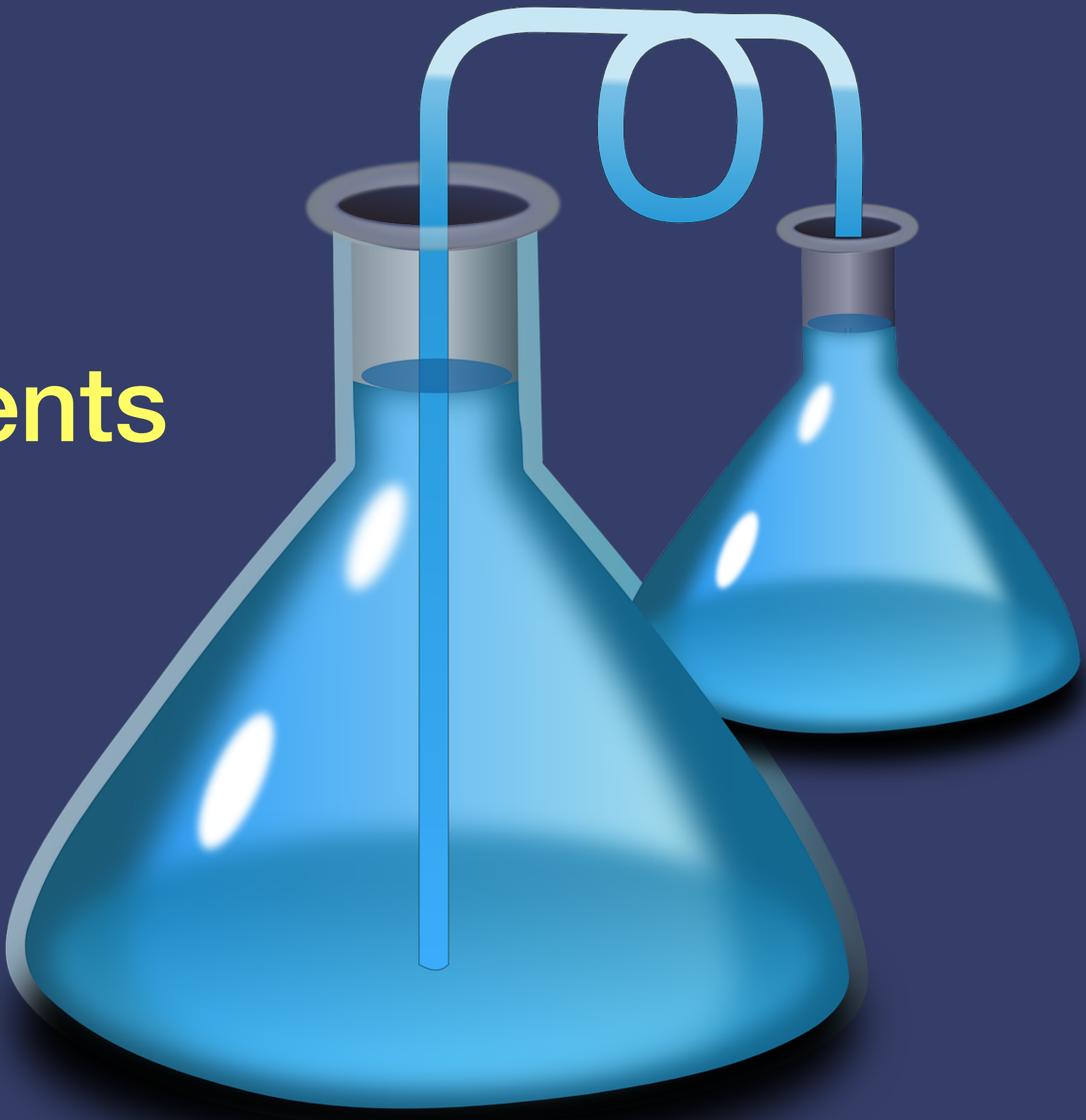
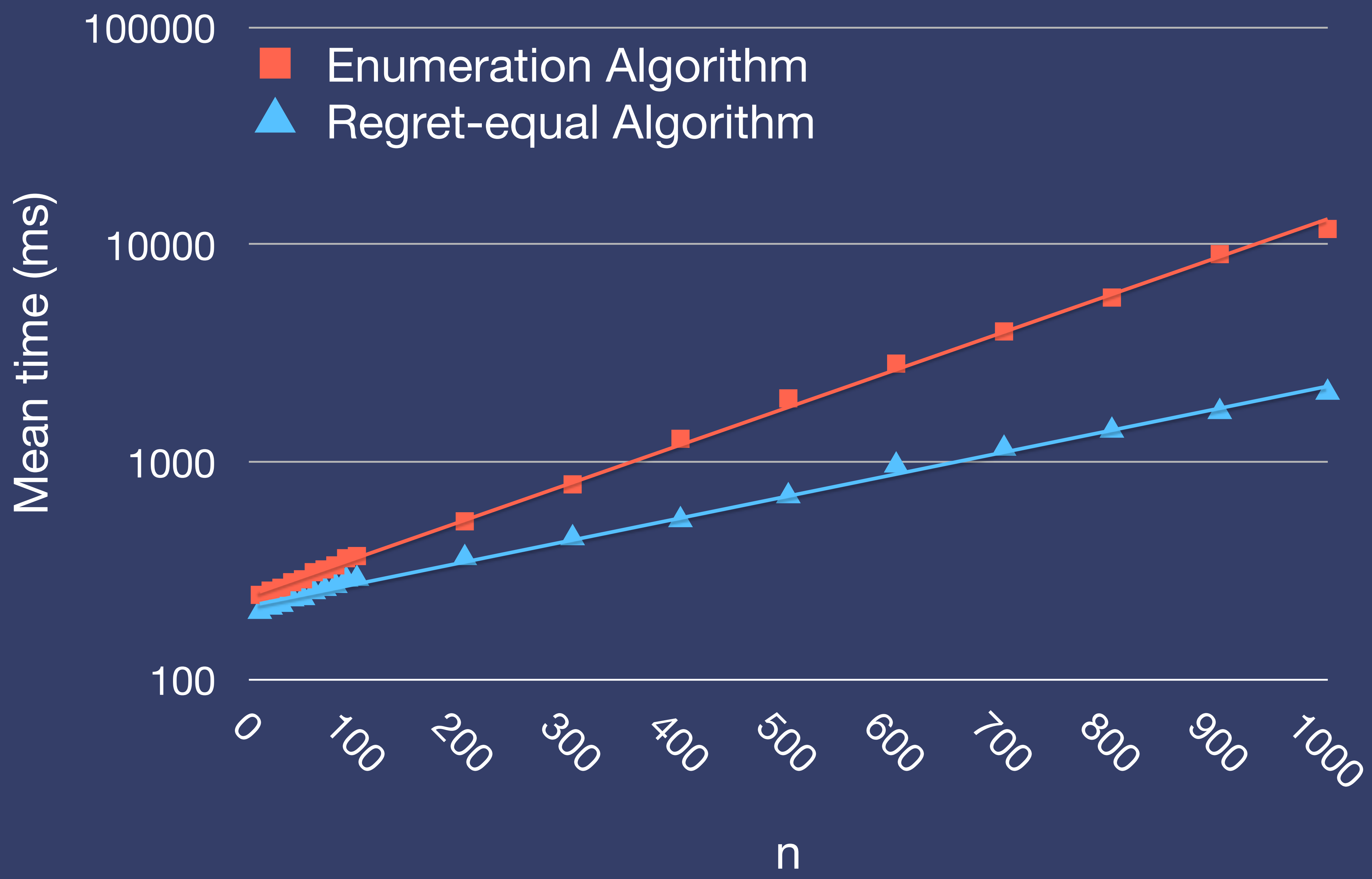
# Experiments



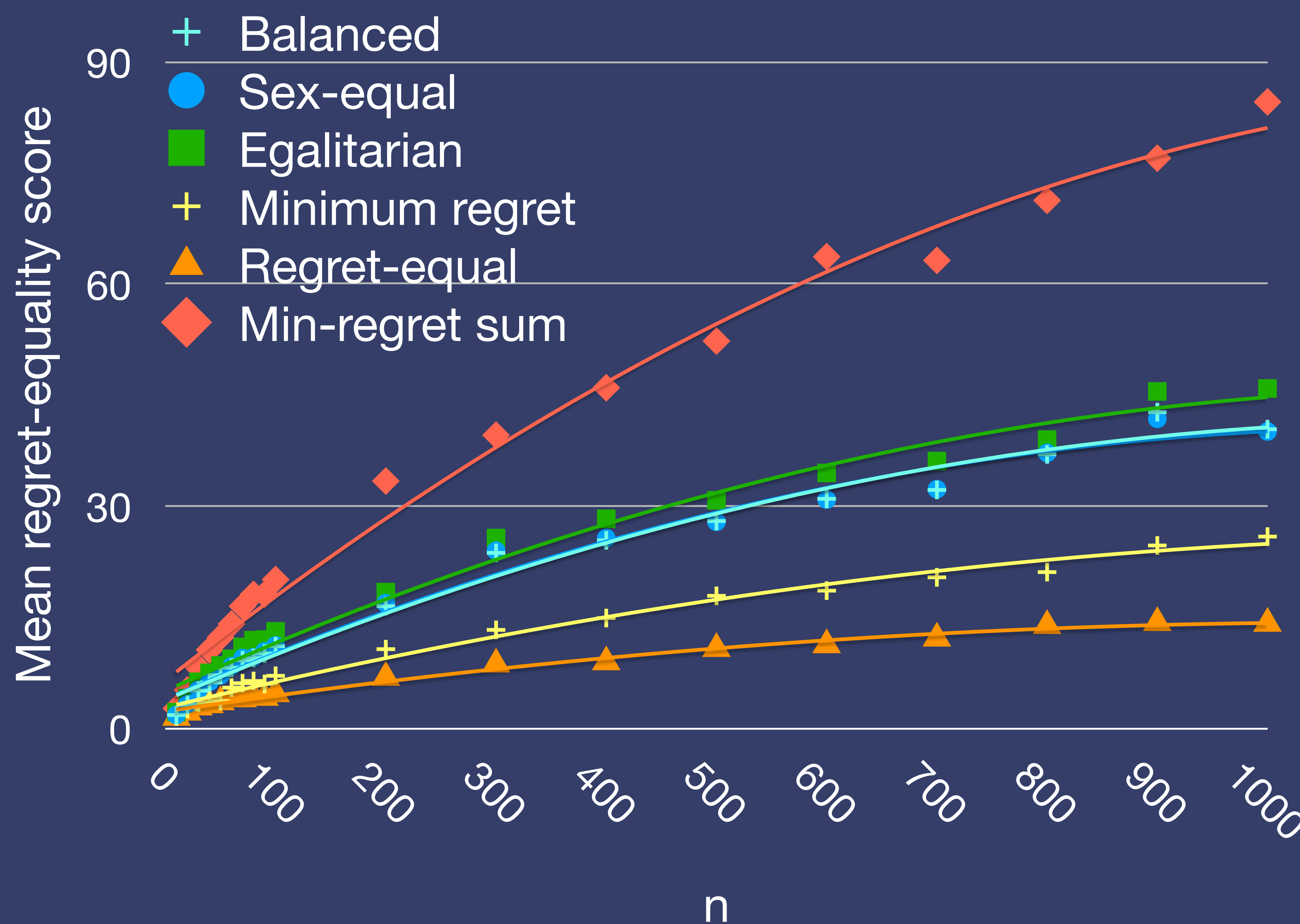
# Methodology

- Performance of the Regret-equal Algorithm compared to an Enumeration algorithm (exponential in worst case)
- Instances size {10, 20, …, 100, 200, …, 1000}, complete preference lists, 500 instance per size.
- looked at properties over several types of optimal stable matching (balanced, sex-equal, egalitarian, minimum regret, regret-equal, min-regret sum)
- Java, Python, Bash, GNU parallel
- Correctness
    - all matchings found were stable
    - Regret-equality scores matched
    - CPLEX up to size n = 50 for the enumeration algorithm

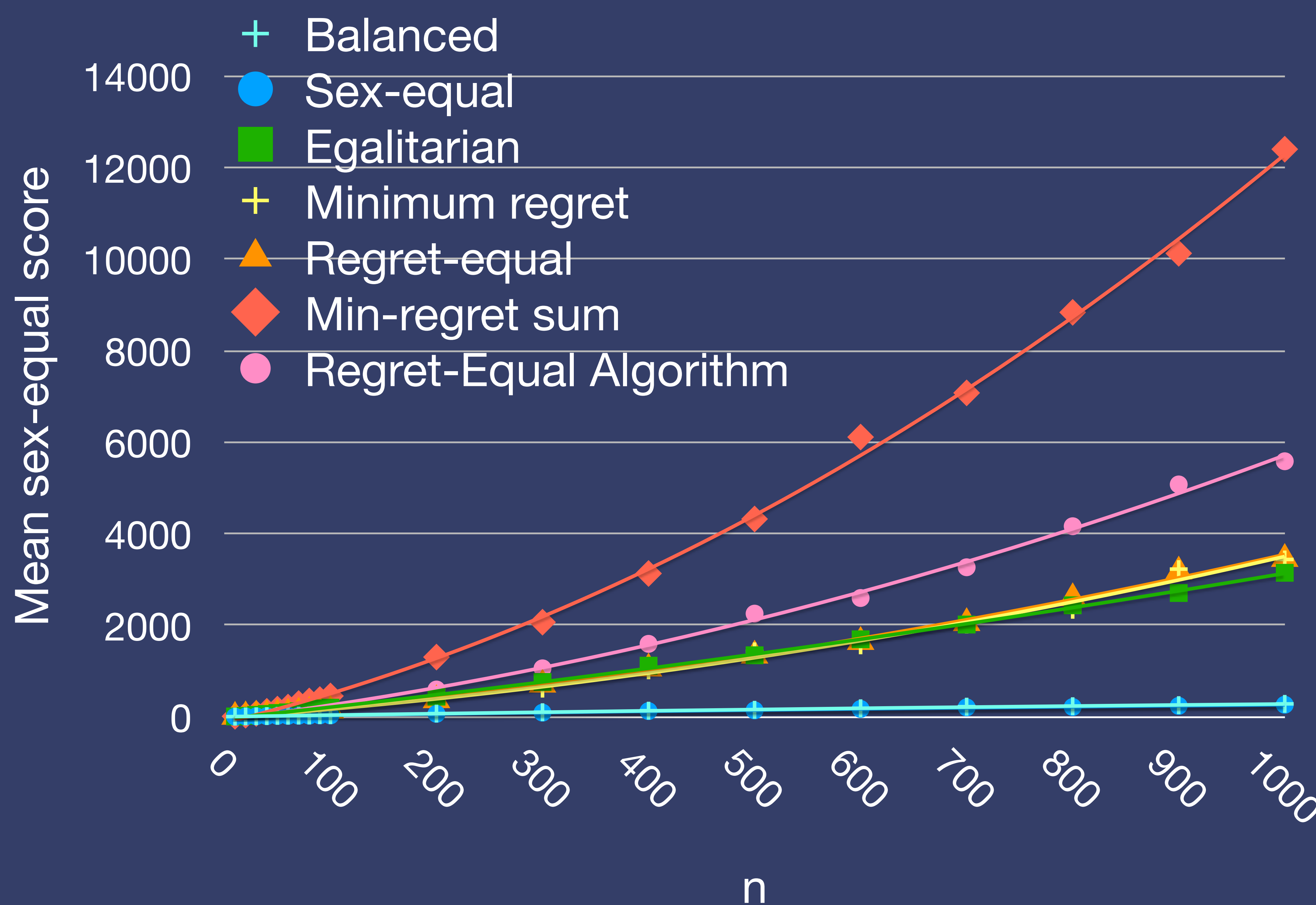
# Time taken


100000
10000
1000
100
Enumeration Algorithm
Regret-equal Algorithm
Mean time (ms)
0
100
200
300
400
500
600
700
800
900
1000
n

# Regret-equality score for different optimal matchings



# Sex-equal score for different optimal matchings



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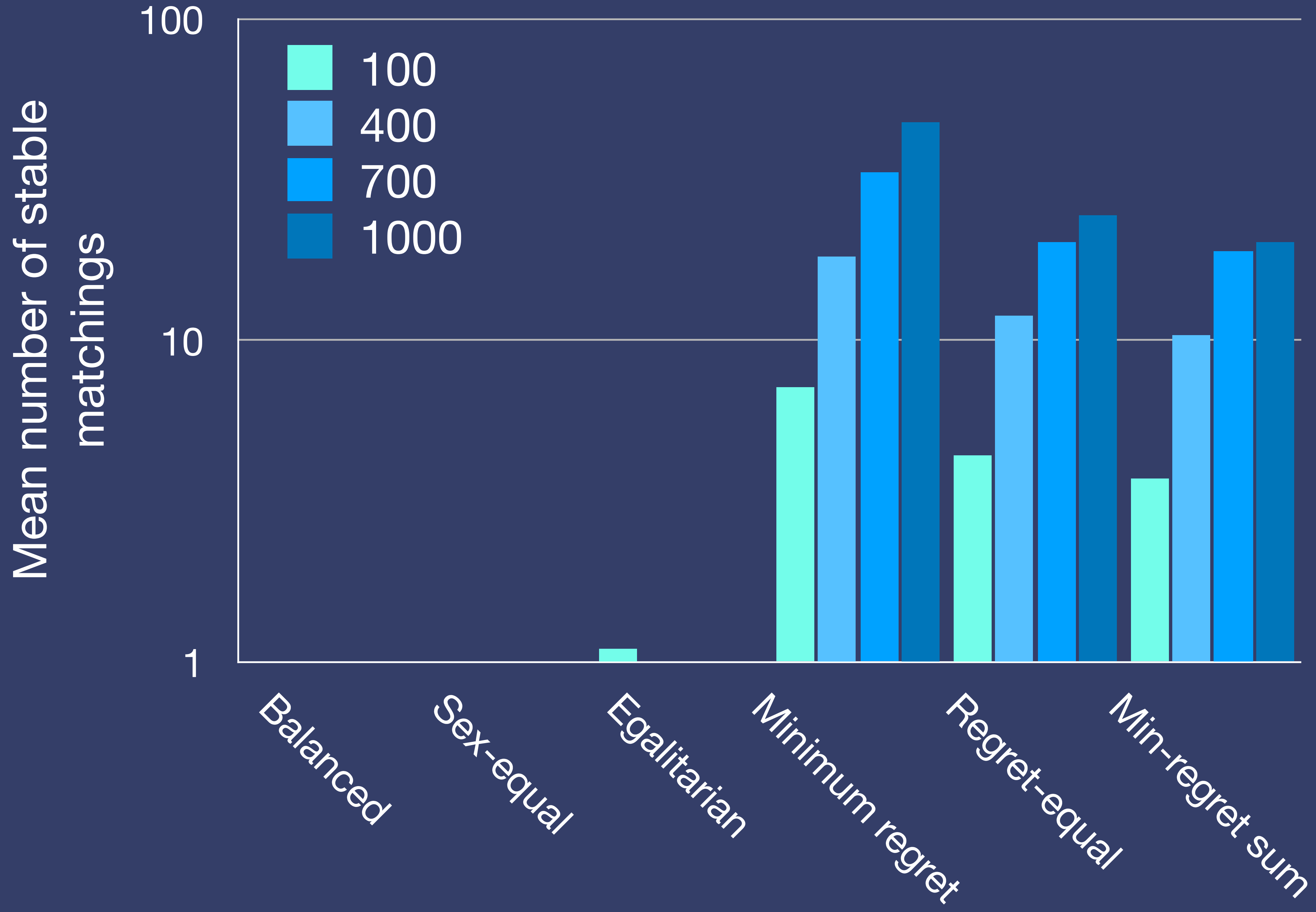
# Frequency of different optimal stable matchings



# Future Work

- Improving the O(nmc) Regret-equal Algorithm, where $c = |d_U(M_0) - d_W(M_0)|$

- Grouping women - e.g. women are workers and men are jobs to assign to workers.

  - Woman optimal stable matching would naturally satisfy ‘balanced’, ‘min-regret', ‘egalitarian’ and ‘min-regret sum’ criteria

  - Can find a ‘regret-equal’ stable matching in $O(n^4)$ time

  - Open problem for ‘sex-equality’ -> grouped-women-equality

# Thank you

## Summary

- Matching problems
- Fairness
- Finding fair stable matchings
- Experiments
- Future work: finding improved algorithms

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