

Profile-based optimal matchings in Stable Marriage

Frances Cooper

Joint work with: Prof David Manlove

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- Matching problems
- Fairness

- Matching problems
- Fairness
- Finding fair stable matchings

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- Experiments

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- Finding fair stable matchings
- Experiments
- Future work









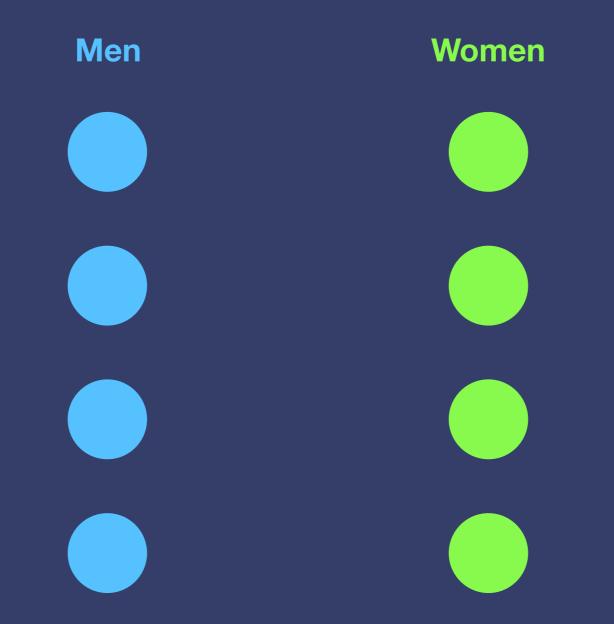


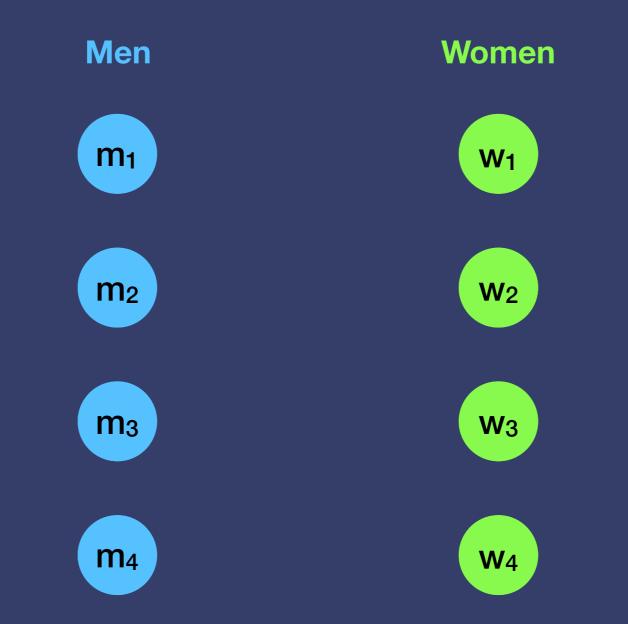
 Assign one set of entities to another set of entities

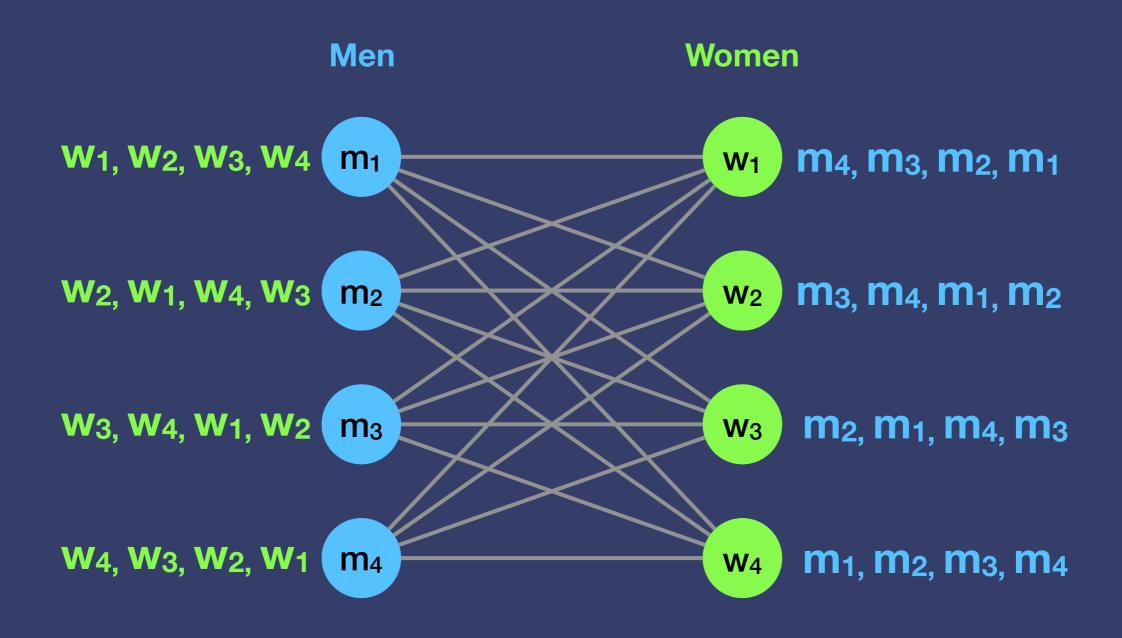


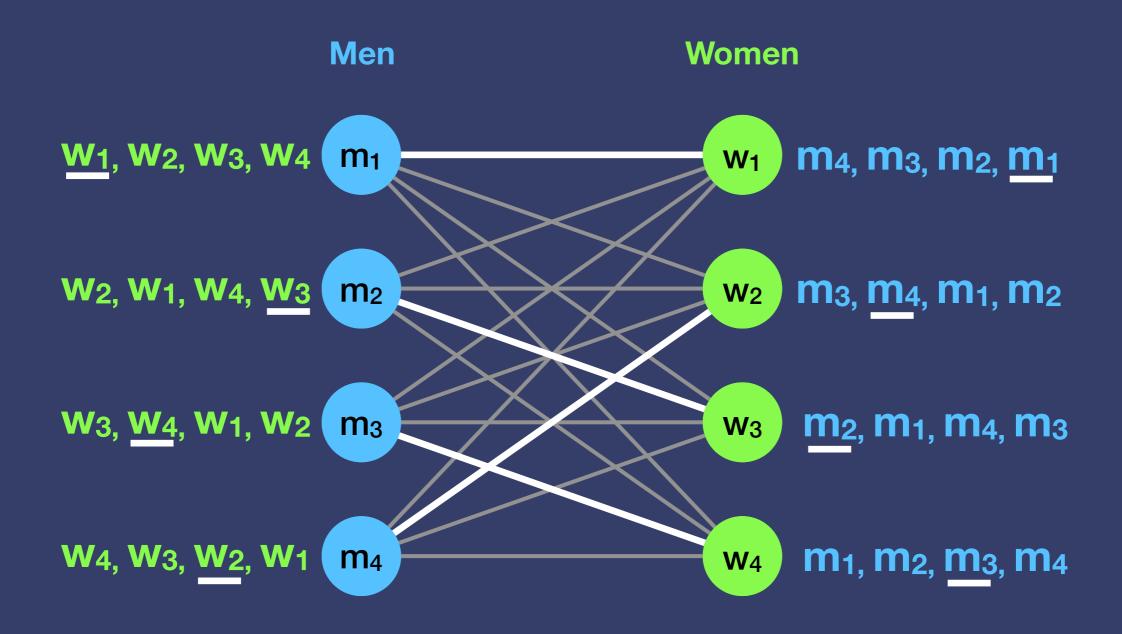


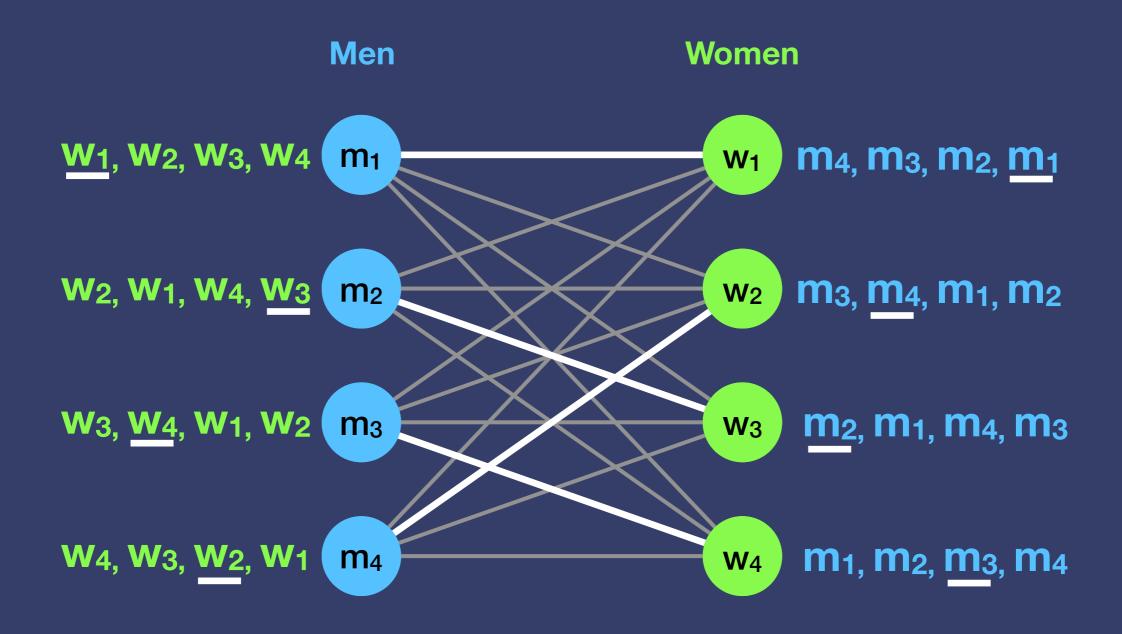
- Assign one set of entities to another set of entities
- Based on preferences and capacities



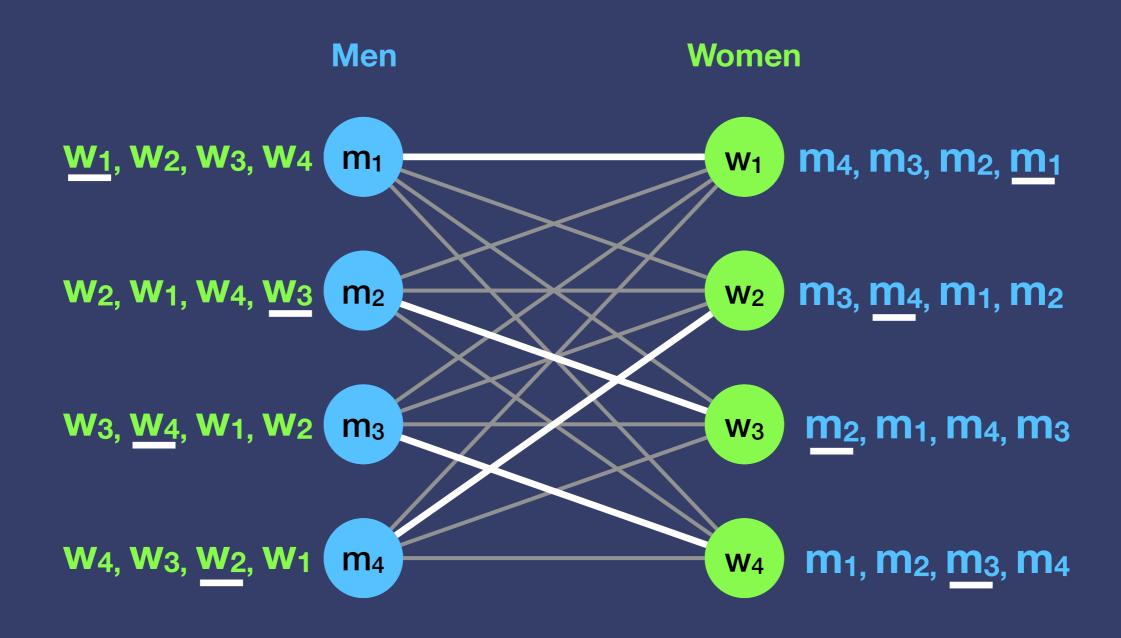








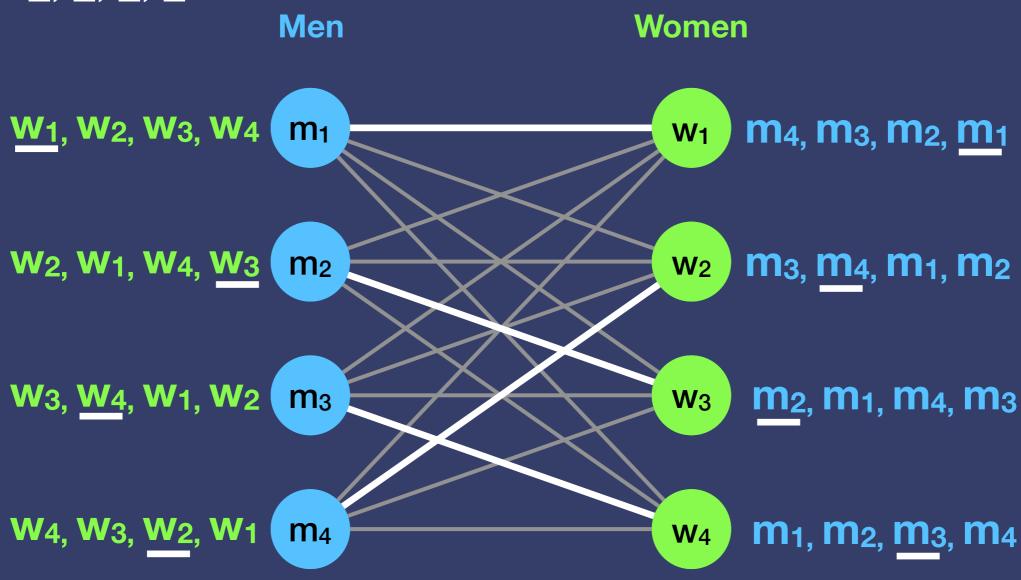
Rank Degree



Stable Marriage

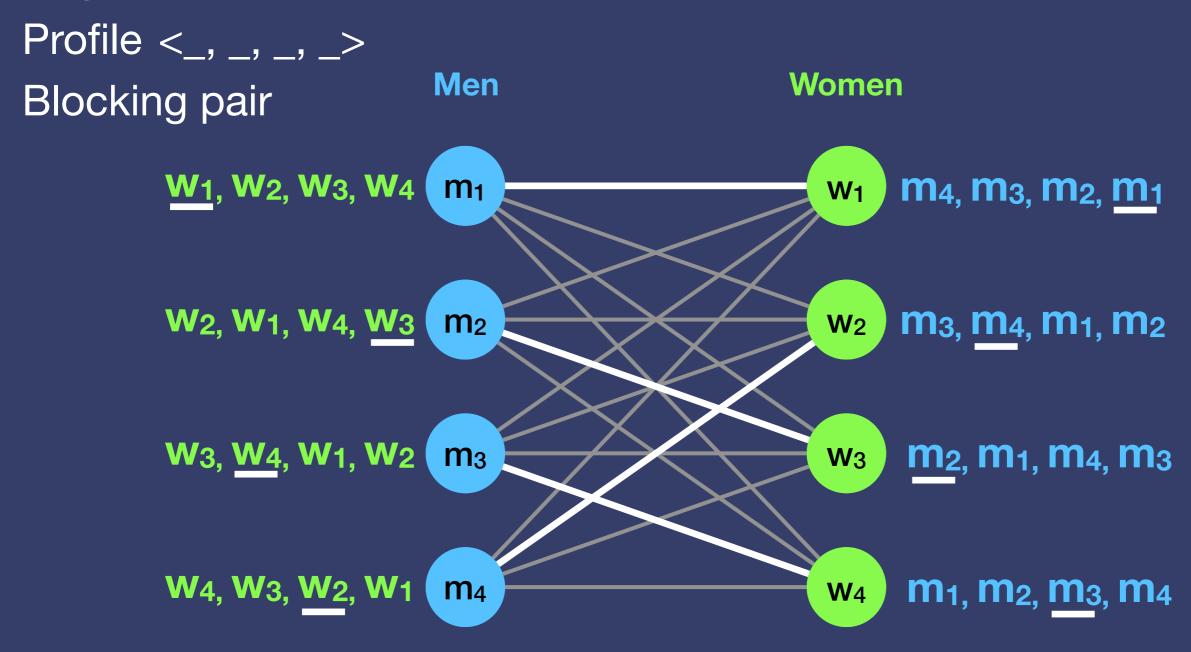
Degree

Profile <_, _, _, _>



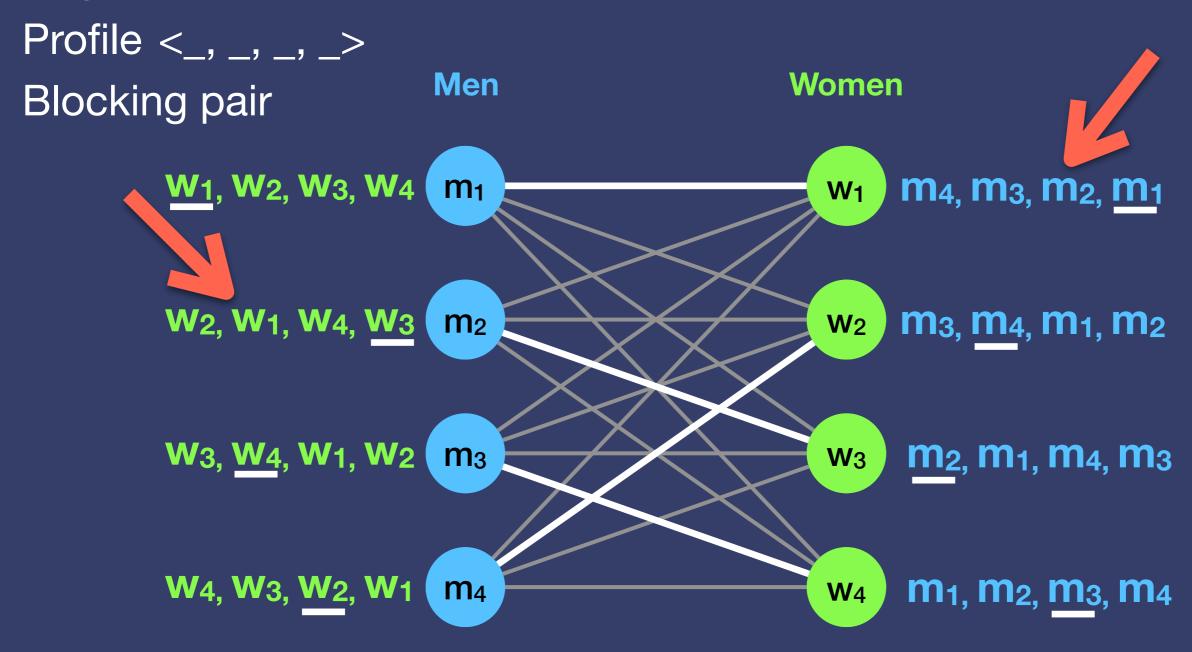
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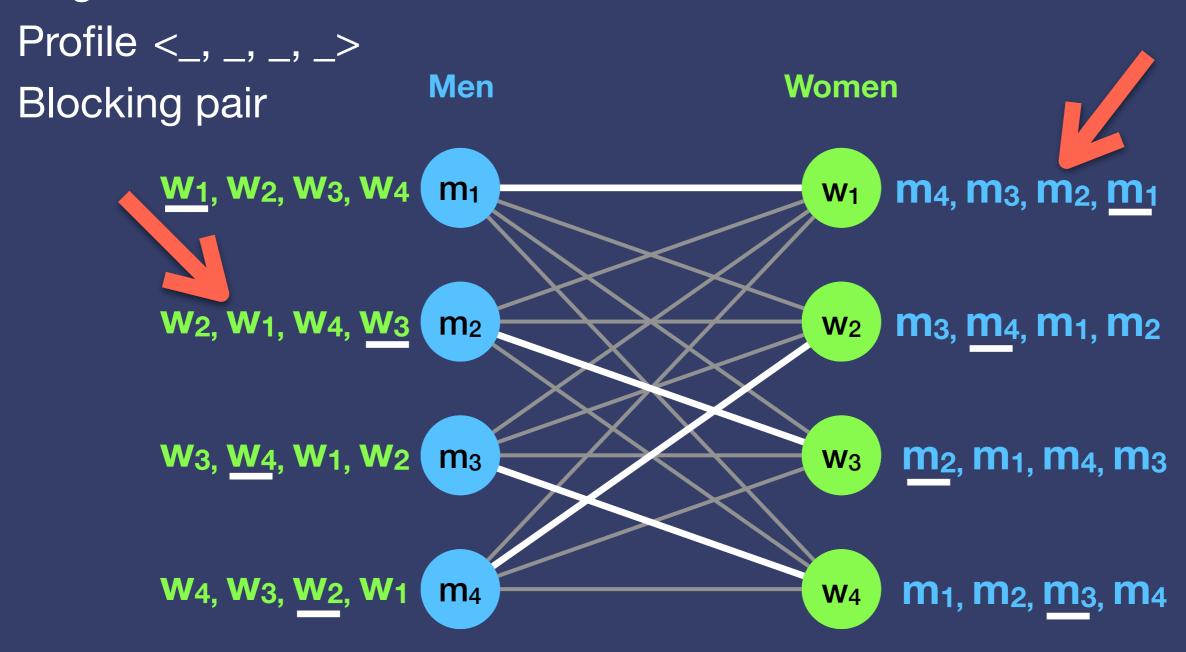
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Stable Marriage

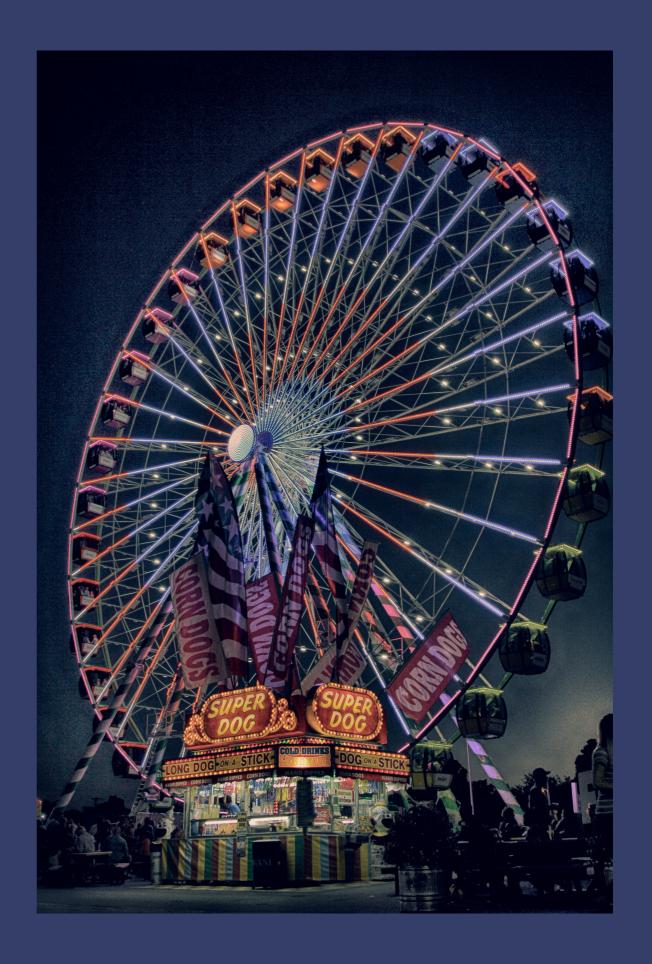
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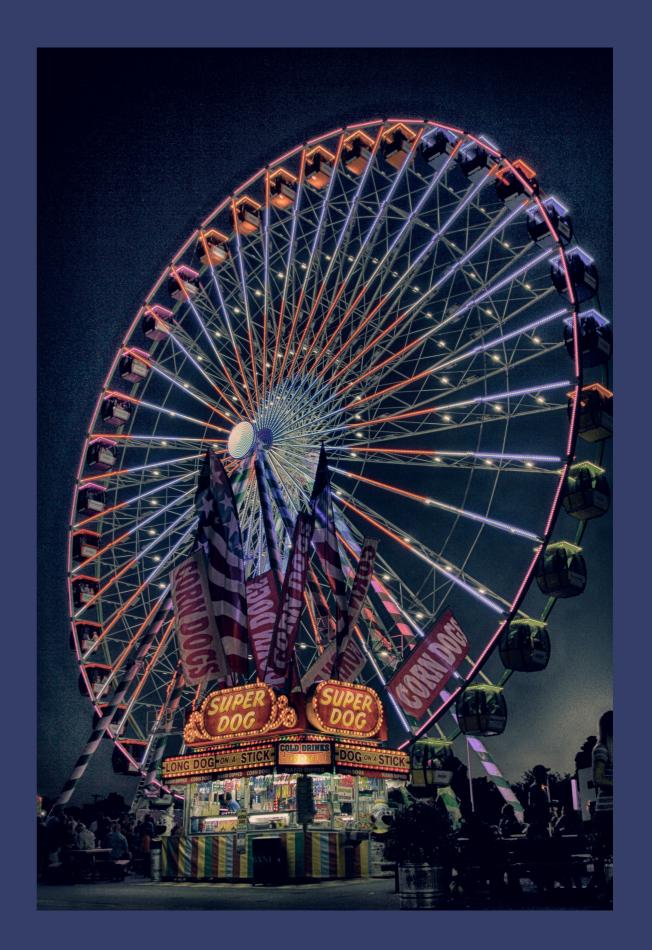
A stable matching is a matching with no blocking pairs



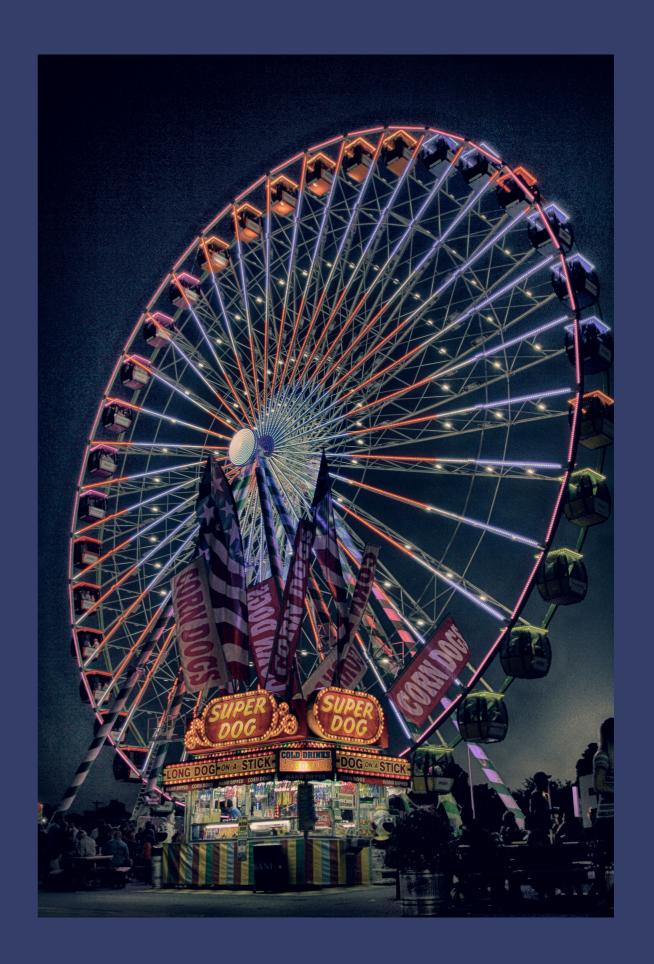
 Many stable matchings per instance



- Many stable matchings per instance
- Want to find a stable matching that provides some kind of equality between men and women



- Many stable matchings per instance
- Want to find a stable matching that provides some kind of equality between men and women
- Several different fairness measures



Sex-equal - minimise the differences between the sum of ranks for men and the sum of ranks for women - NP-hard

Complexity of the Sex-Equal Stable Marriage Problem; Japan Journal of Industrial and Applied Mathematics; 1993; Kato

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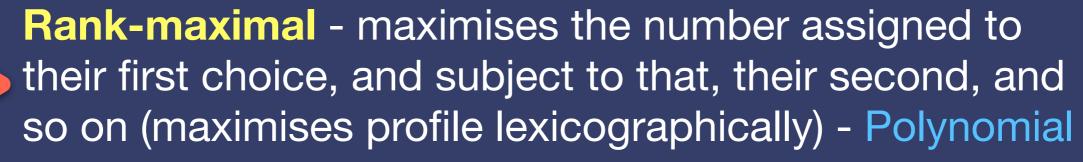
Rank-maximal - maximises the number assigned to their first choice, and subject to that, their second, and so on (maximises profile lexicographically) - Polynomial

Generous - minimises the number assigned to their last choice, and subject to that, their second to last, and so on (minimises reverse profile lexicographically) - Polynomial

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Sex-equal

Sum men ranks: 10

Sum women ranks: 10

Sex-equal difference: 0

Profile: <0, 4, 4, 0>

```
      m1: W1, W2, W3, W4
      W1: m4, m3, m2, m1

      m2: W2, W1, W4, W3
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      m3: W3, W4, W1, W2
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Sum women ranks: 10

Sex-equal difference: 0

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Rank-maximal

Sum men ranks: 4

Sum women ranks: 16

Sex-equal difference: 12

Profile: <4, 0, 0, 4>

m₁: W₁, W₂, W₃, W₄ m₂: W₂, W₁, W₄, W₃

m₃: **w**₃, **w**₄, **w**₁, **w**₂

M4: W4, W3, W2, W1

w₁: m₄, m₃, m₂, m₁

w₂: m₃, m₄, m₁, m₂

W3: m2, **m**1, **m**4, **m**3

W4: m₁, **m**₂, **m**₃, **m**₄

m₁: <u>W</u>₁, W₂, W₃, W₄

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Fairness measures

Sex-equal

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m4: W4, W3, <u>W2</u>, W1

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women ranks: 10 Sex-equal difference: 0 Profile: <0, 4, 4 Rank-maximal

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Finding a Rank-maximal stable matching

Using O(n⁵log n)
Irving, Gusfield and
Leather Approach

An efficient algorithm for the "optimal" stable marriage; Journal of the ACM; 1987; Irving, Gusfield, Leather

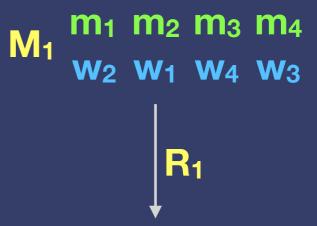


```
R_1 M_2 M_3
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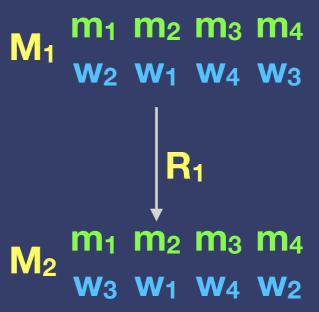
```
R_1 M_1 M_4 M_2 M_3
```

```
M<sub>1</sub> m<sub>1</sub> m<sub>2</sub> m<sub>3</sub> m<sub>4</sub> w<sub>2</sub> w<sub>1</sub> w<sub>4</sub> w<sub>3</sub>
```

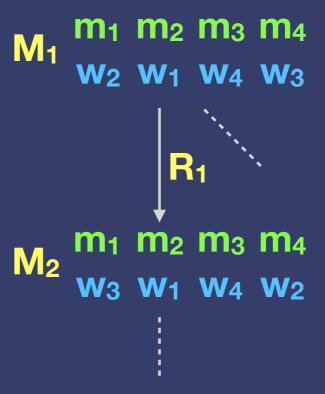






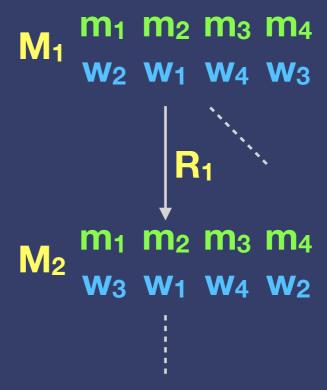






 Rotation - series of man-woman pairs that take us from one stable matching to another when permuted

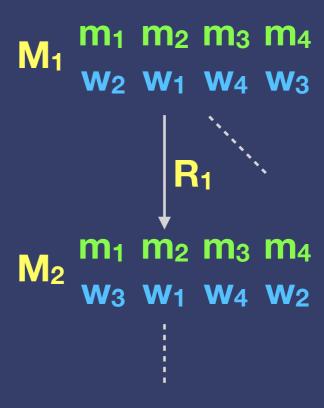
Can only eliminate exposed rotations



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```
R_2 \begin{array}{c} m_1 m_2 \\ w_1 w_2 \end{array}
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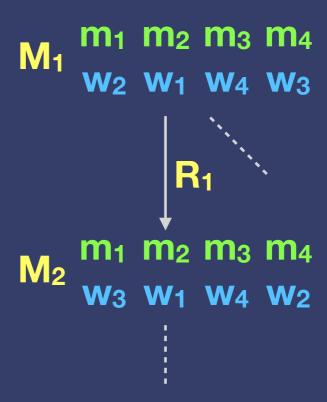


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O(n²) algorithm to find all rotations



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$$w(p) = p_1 * n^{n-1} + p_2 * n^{n-2} + ... + p_n$$
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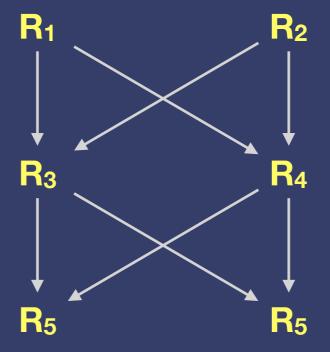
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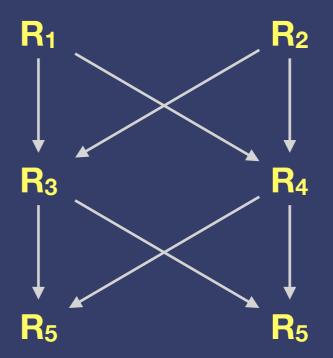
If a rotation has a positive weight then we want to eliminate it if possible as it helps us find a rank-maximal matching.

Displays order in which rotations can be eliminated

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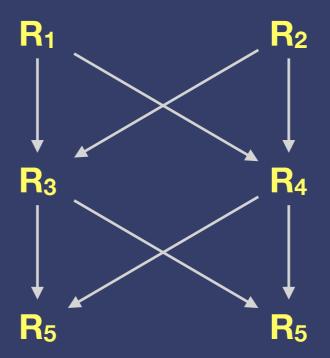


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 correspondence with the closed subsets of the poset

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- The set of stable matchings are in 1-1
 correspondence with the closed subsets of the poset
- Want: Max weight closed subset of the rotation poset

Build a flow network based on the rotation poset

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can be improved - _____we'll look at this later

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O(n5log n) using Irving, Gusfield and Leather approach

Weights are exponentially large:

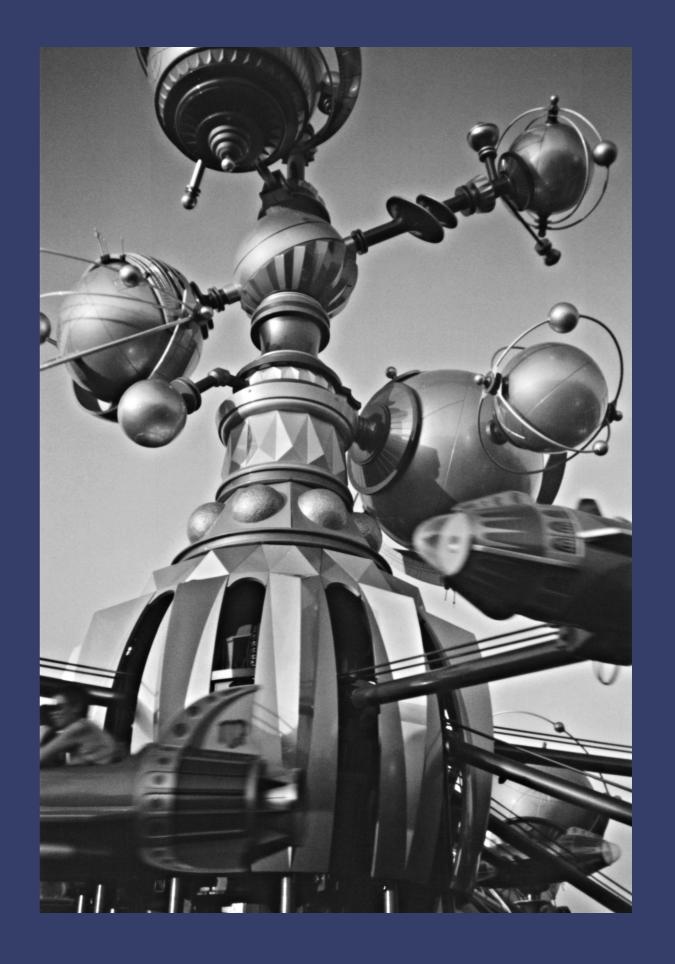
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- Calculations may cause overflow / inaccuracies for primitive types
- Memory issues for types that can store arbitrarily large numbers

Rank-maximal stable matching using a vector-based approach



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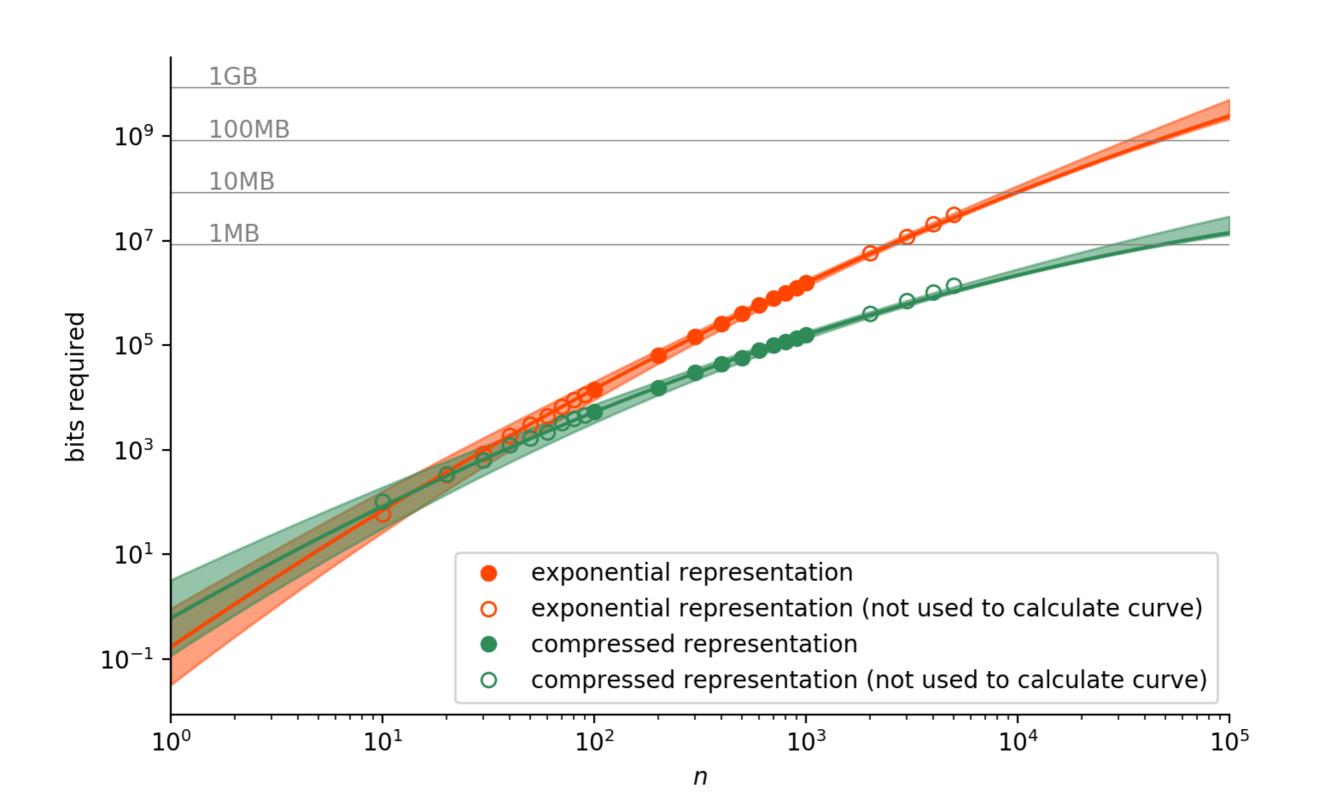
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$$<1, 4, 0, 0, -2, 0, 0, 0, 0, 0, 0, -3, 0, 0, 0, 0 >$$



save the index and value of non zero elements (lossless)

$$<(0,1), (1,4), (4,-2), (11,-3)>$$



Steps

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- Had to define our own arithmetic over these vectors

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Can find a rank-maximal stable matching in O(n⁵ log n) using vectors - matches the exponential approach (but with added bonus of vector compression)



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Generous stable matching can be found in O(n²d³ log n) time - competitive when d is small d is degree of the minimum-regret stable matching





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all stable matchings

Java, Python, Bash, GNU parallel

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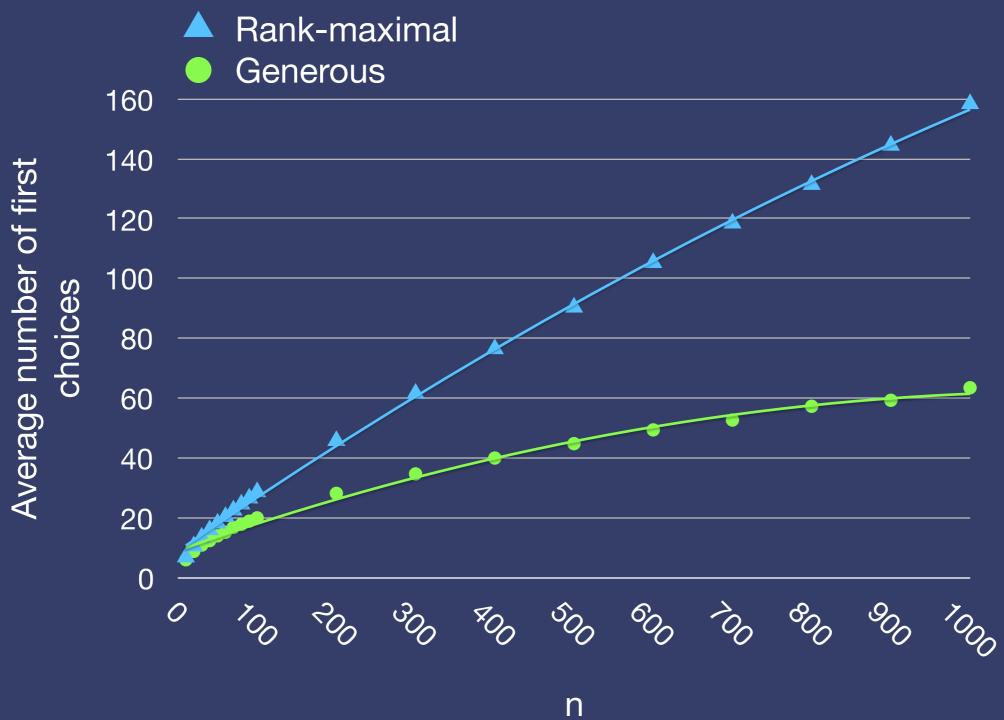
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- Java, Python, Bash, GNU parallel
- Correctness
 - all stable matchings found were stable
 - CPLEX up to size n = 60 for the number of stable matchings

Average number of first choices

Average number of first choices



Average degree

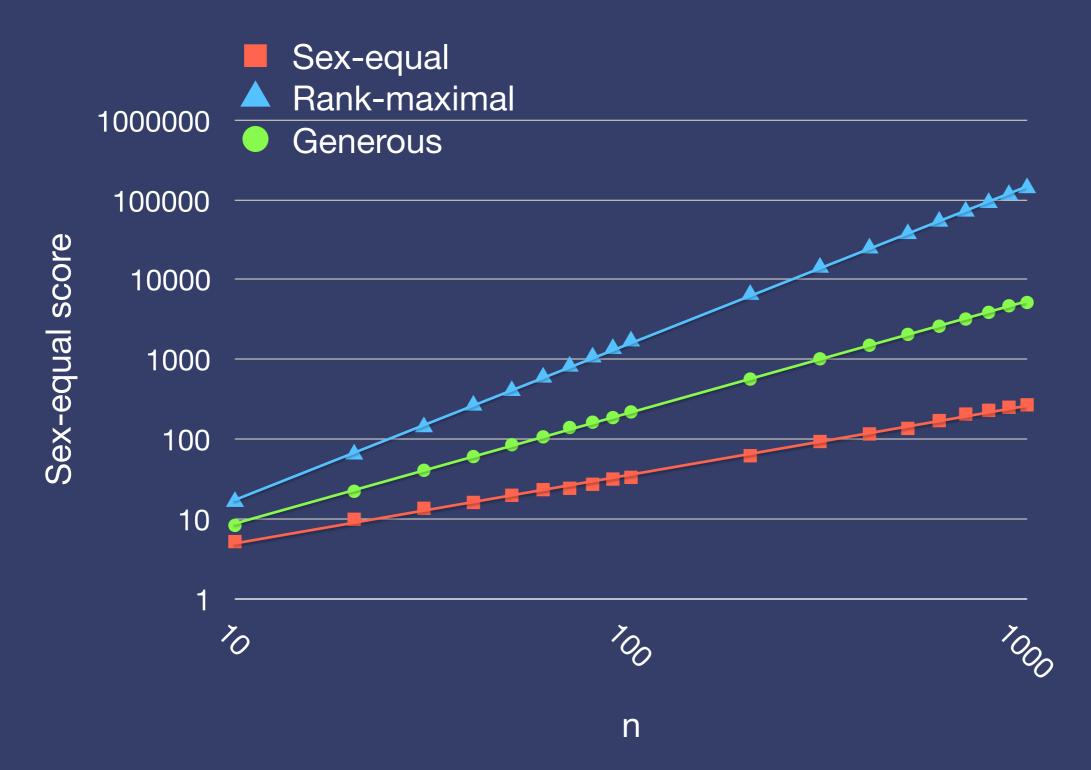
Average degree



25

Sex-equal score

Sex-equal score



Adapt Orlin's Max Flow algorithm

Max Flows in O(nm) Time, or Better; Association for Computing Machinery; 2013; Orlin

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Based on weighted SAT

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Max Flows in O(nm) Time, or Better; Association for Computing Machinery; 2013; Orlin

- Would get O(n⁵)
 - improvement automatically available in the exponential case
- Adapt Feder's technique
 - Based on weighted SAT
 - Would get O(n^{4.5})

A new fixed point approach for stable networks and stable marriages; Journal of Computer and System Sciences; 1992; Feder

Thank you

Summary

- Matching problems
- Fairness
- Finding fair stable matchings
- Experiments
- Future work: adapting algorithms to vector-based setting for improved time complexity



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