A 3/2-approximation algorithm for the student-project allocation problem with ties

Frances Cooper

Joint work with: Dr David Manlove

Outline

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- Matching problems

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- Maximum sized stable matching

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    - Integer programming

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- Future work

Matching Problems

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- Assign one set of entities to another set of entities

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- Assign one set of entities to another set of entities
- Based on preferences and capacities

Student-project allocation problem (SPA-ST)

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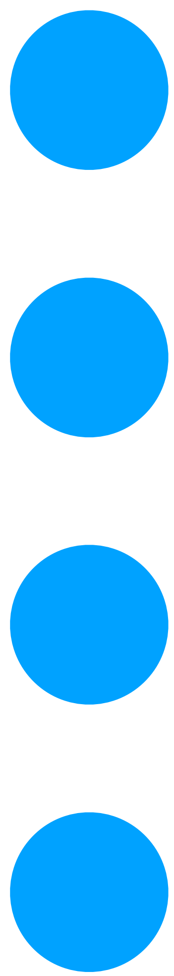
Students



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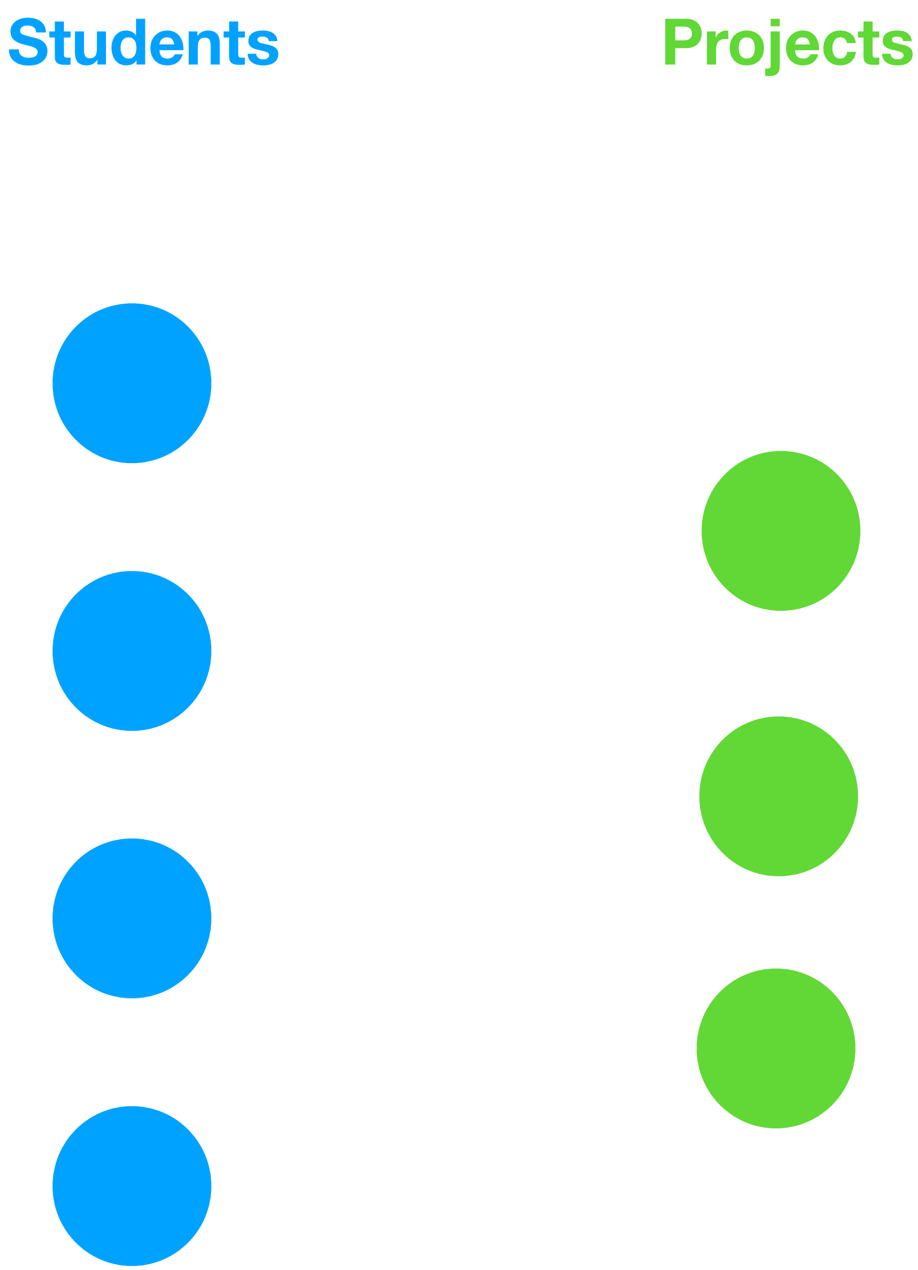

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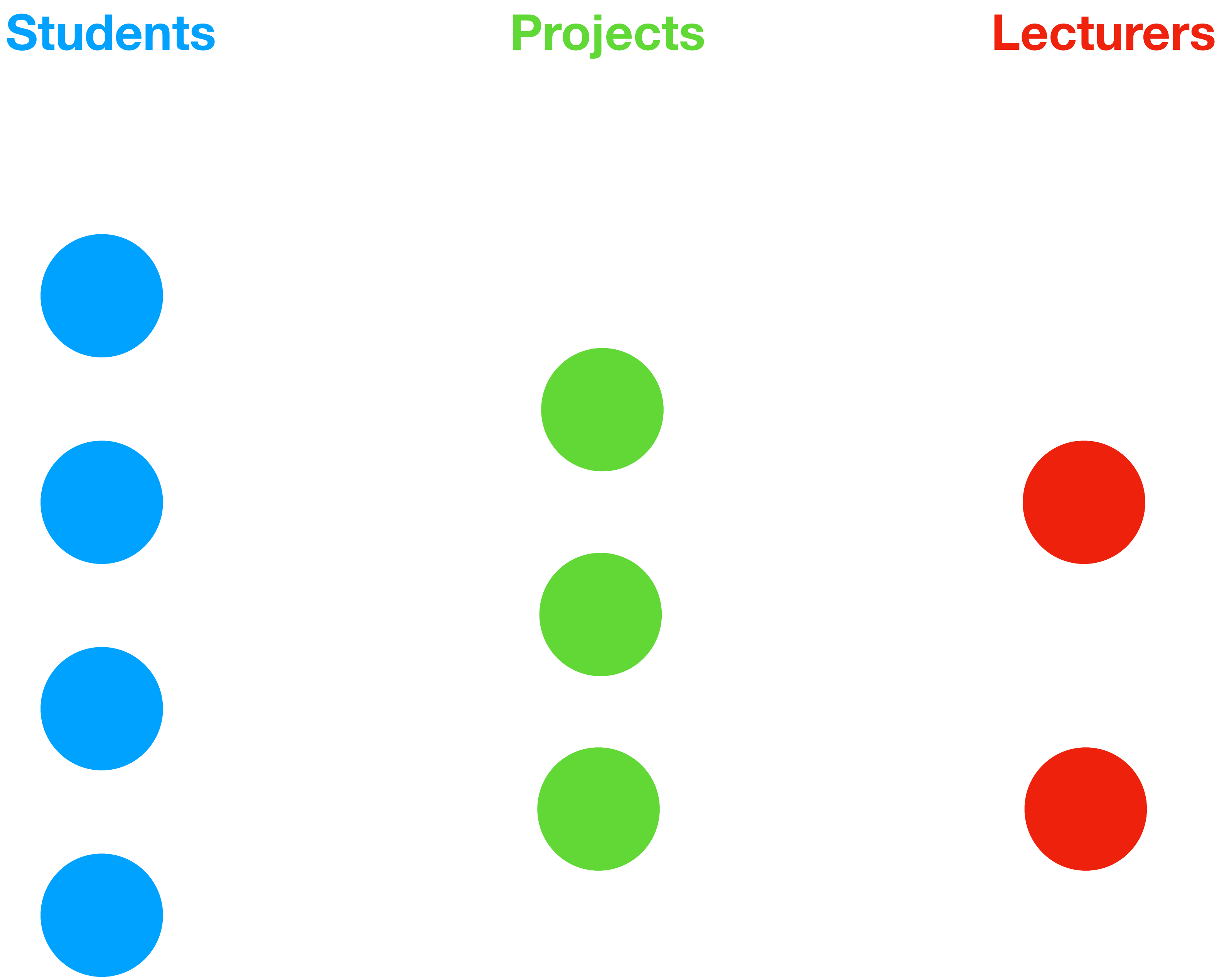
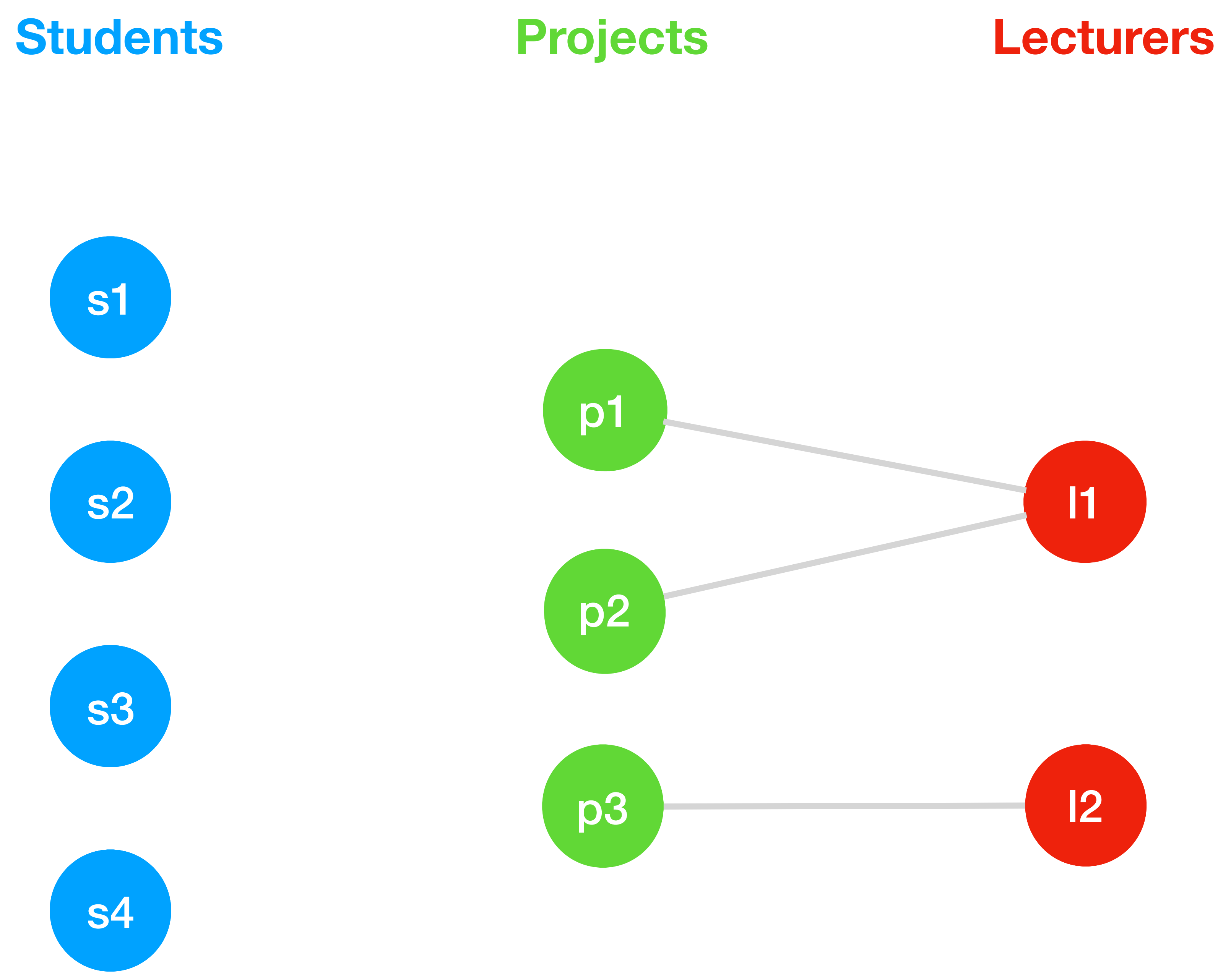


Student-project allocation problem (SPA-ST)

Students **Projects** **Lecturers**

Student-project allocation problem (SPA-ST)



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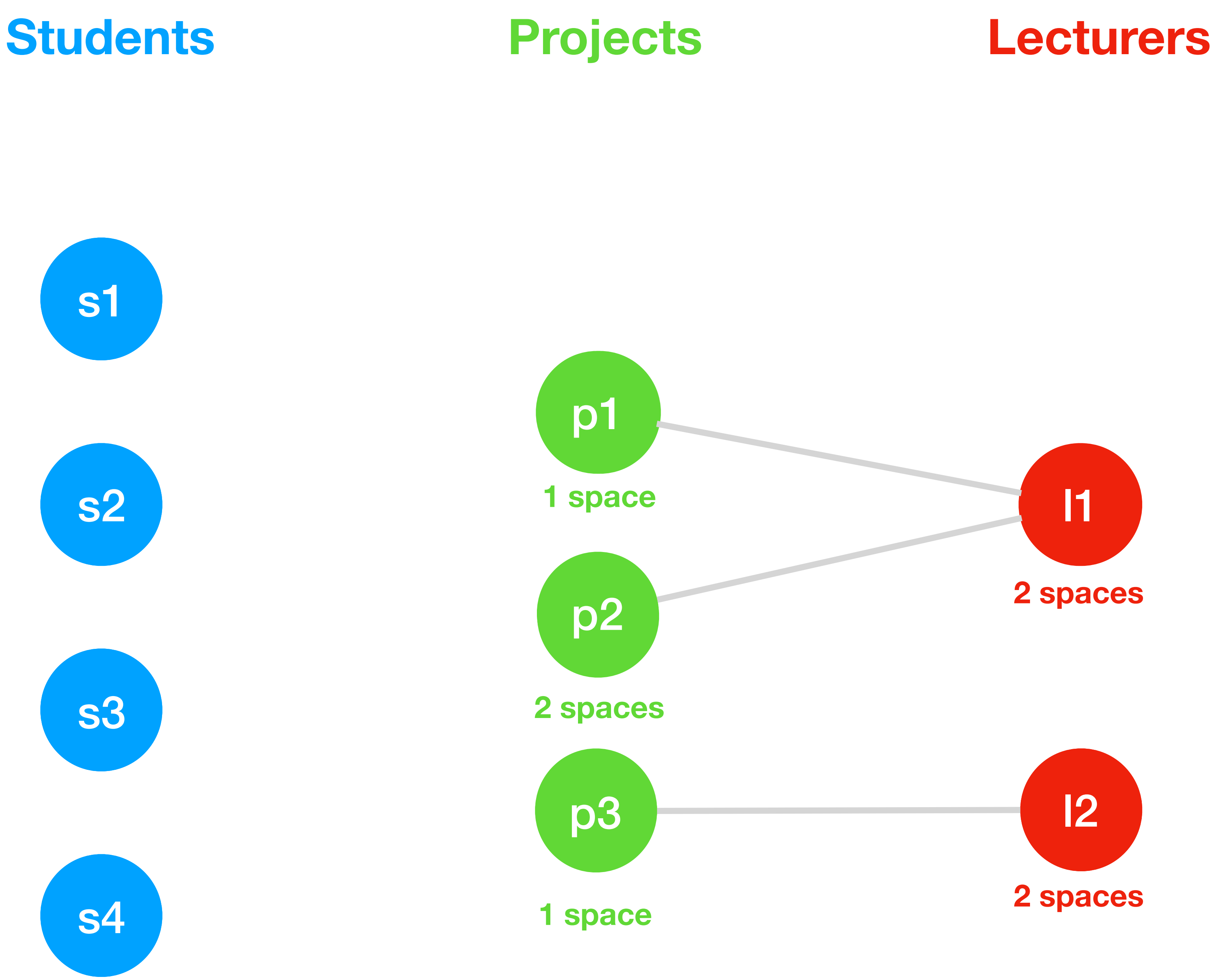

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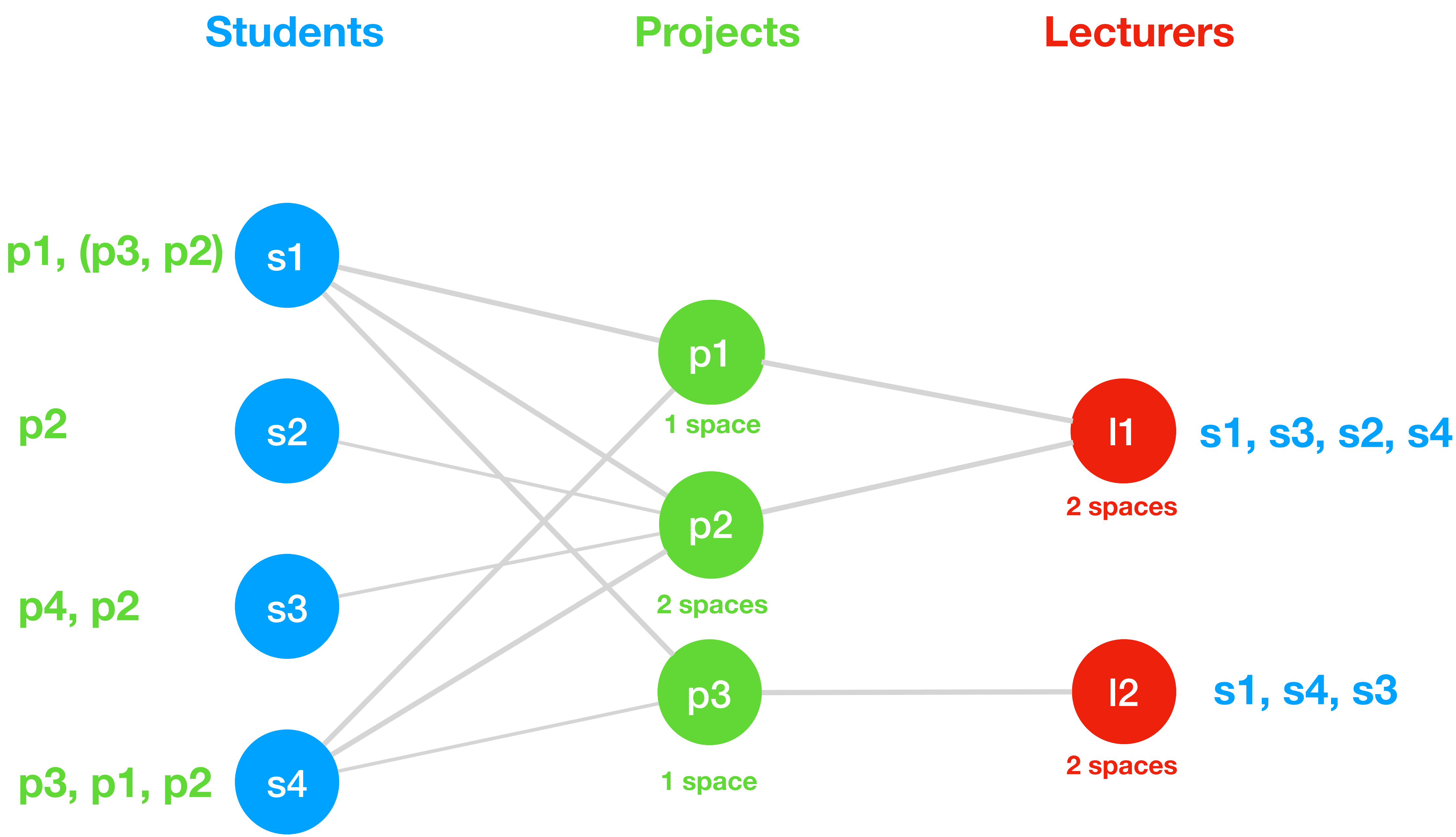

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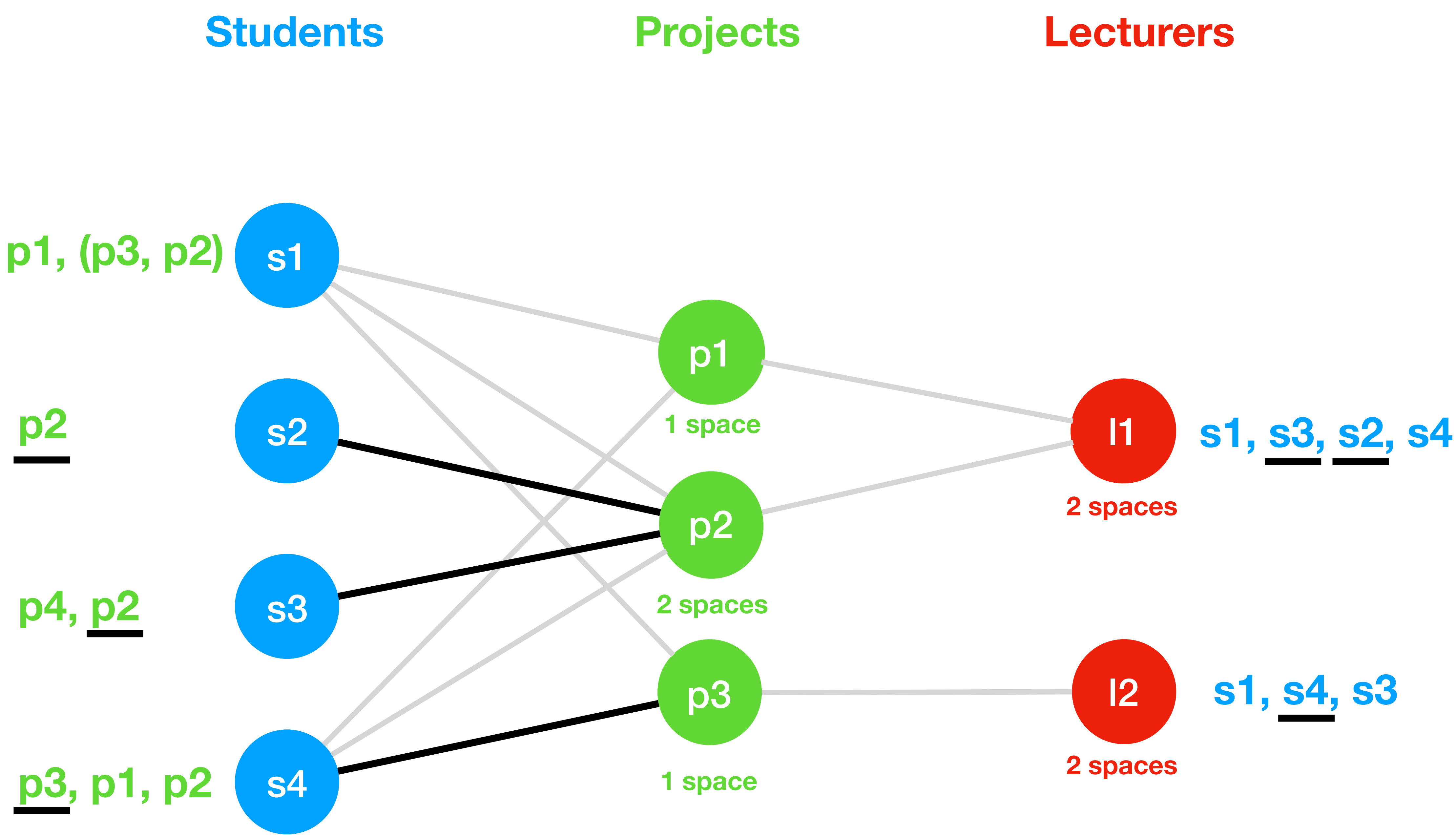

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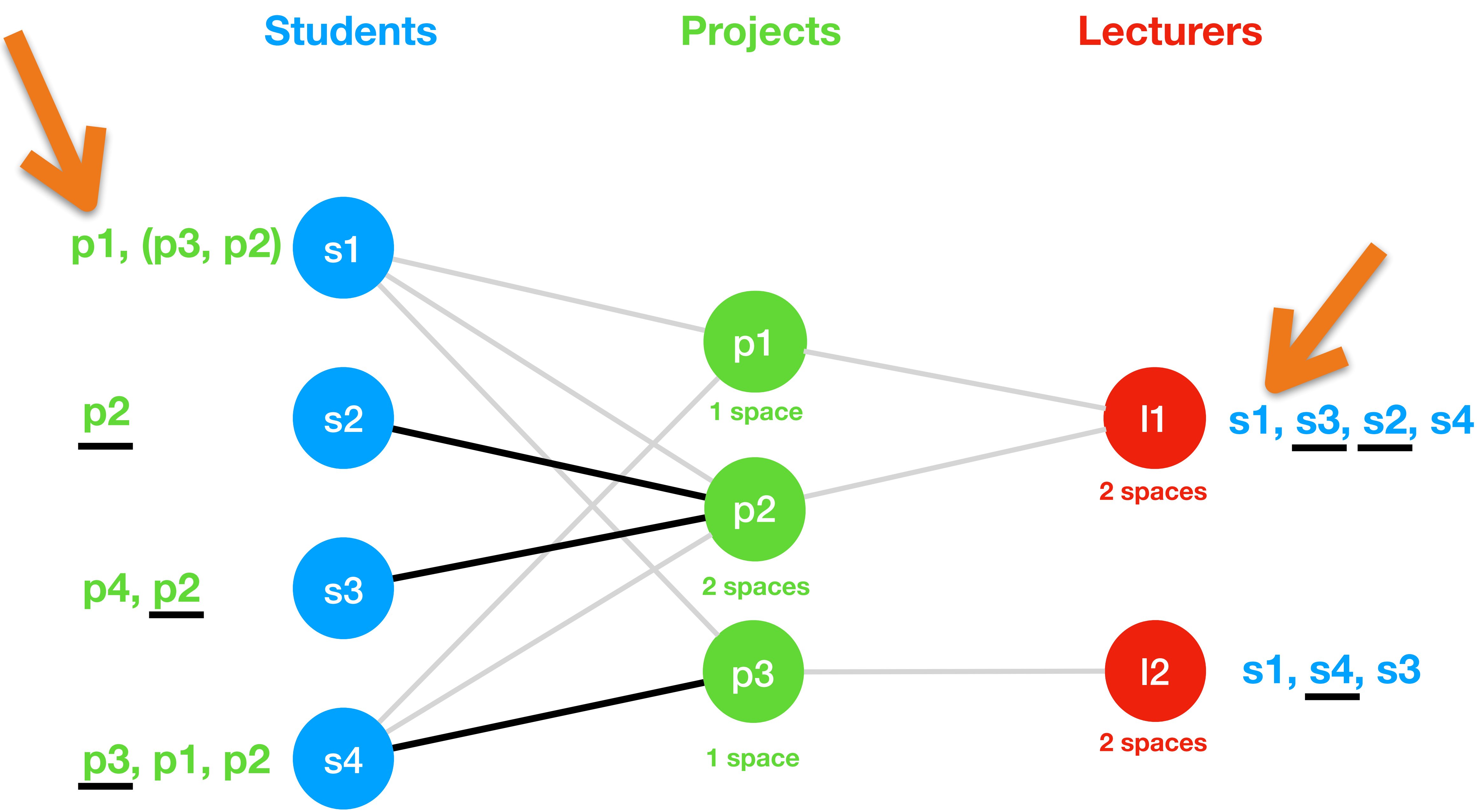
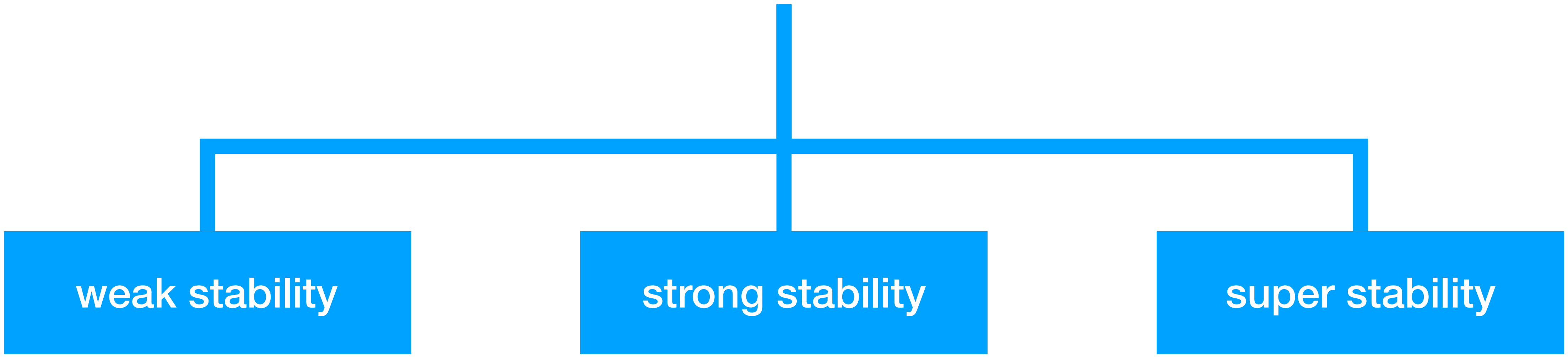


Stable matching

Stable matching



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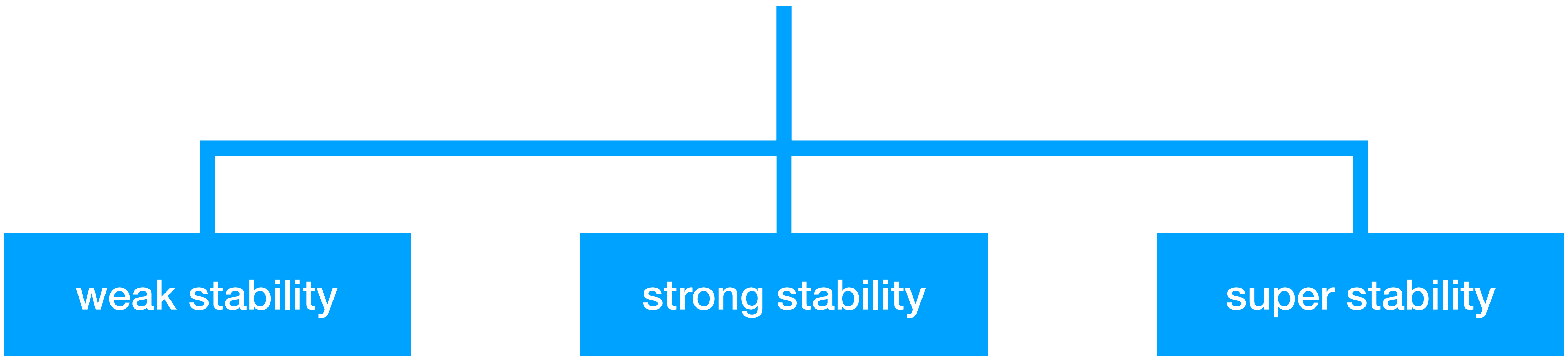

- A **stable matching** is a matching with no blocking pairs

Stable matching

weak stability

No Blocking pairs: both agents are better off

strong stability

No Blocking Pairs: one agent is better off, the other is no worse off

super stability

No Blocking Pairs: neither agent is worse off

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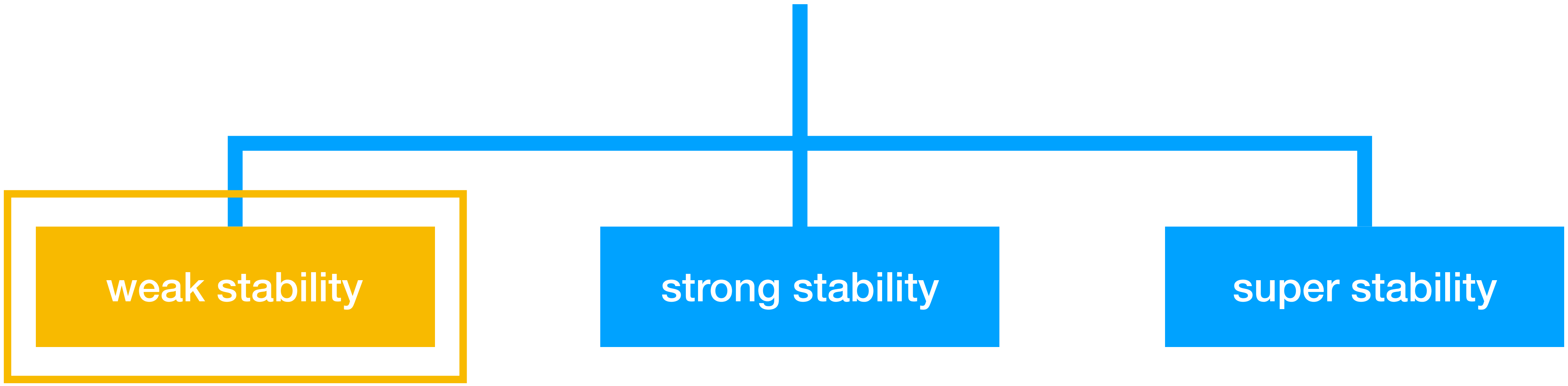

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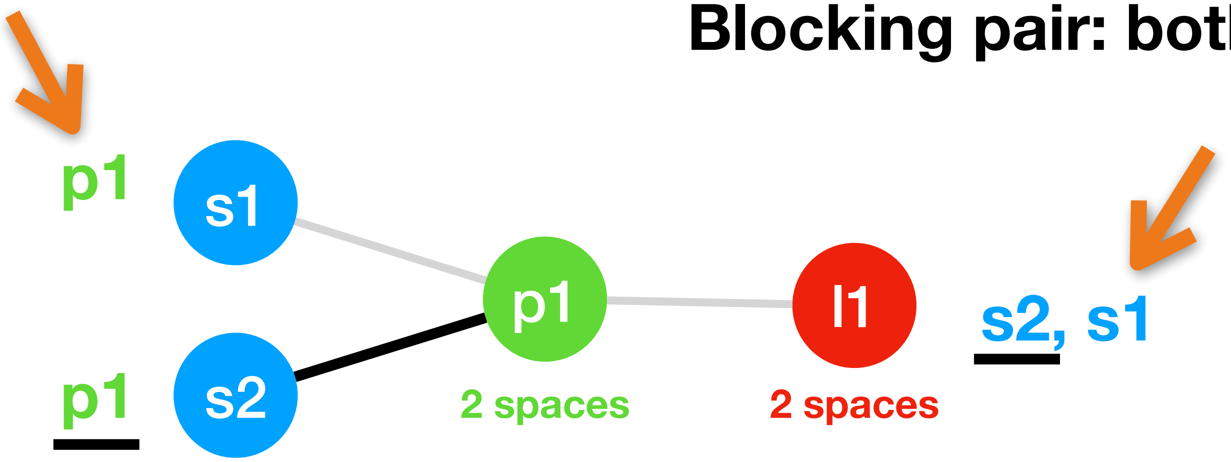

project and lecturer undersubscribed

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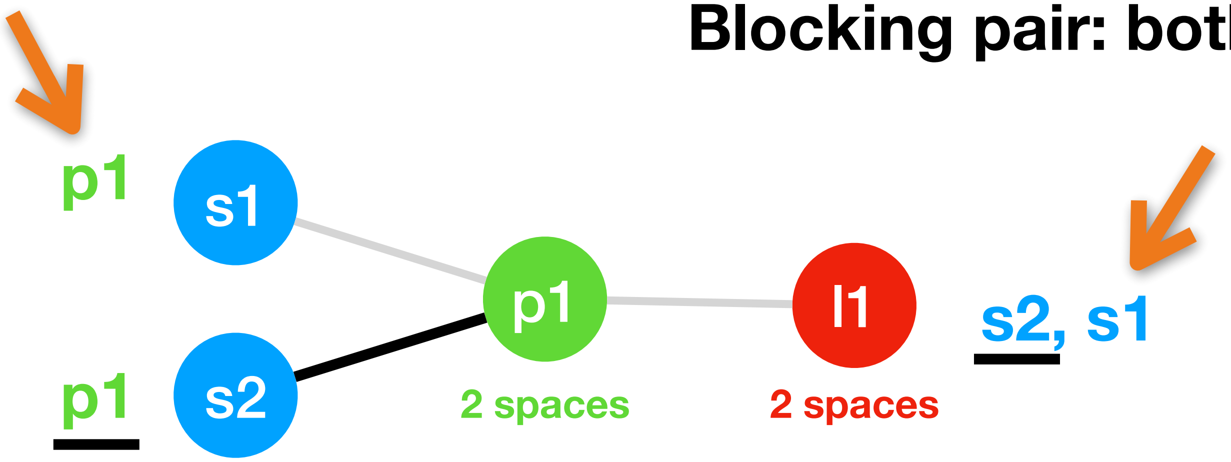

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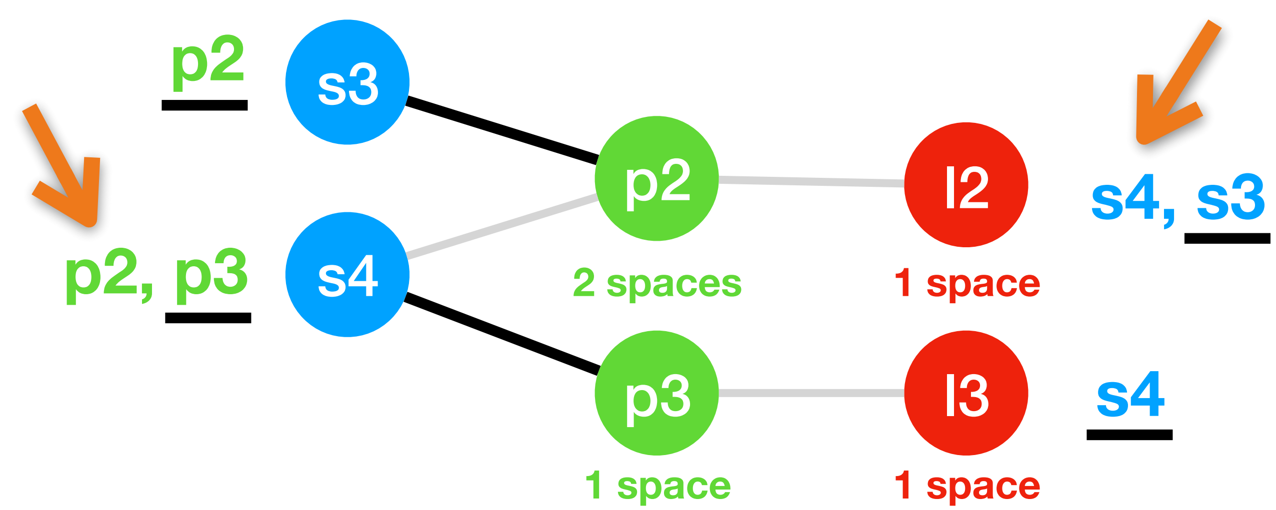

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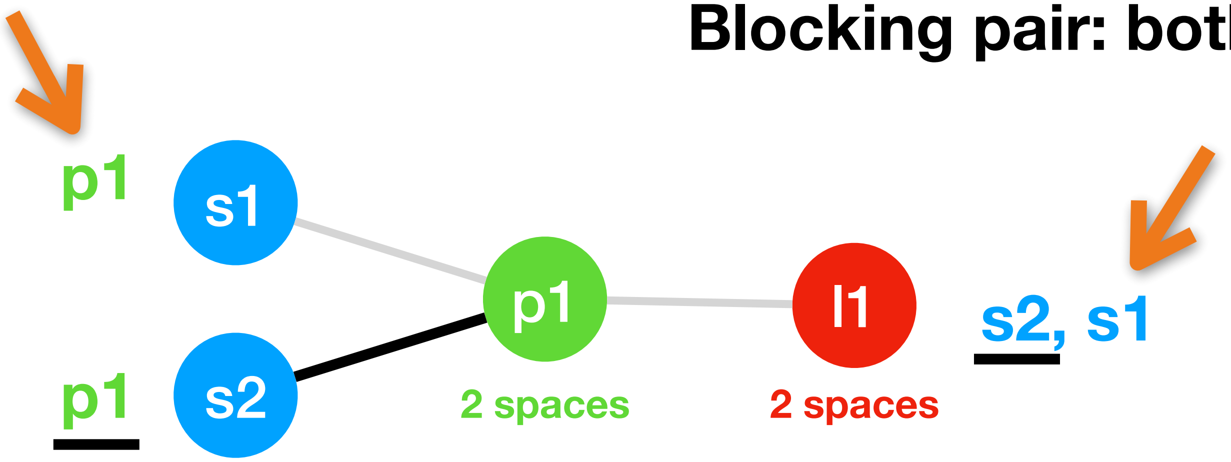

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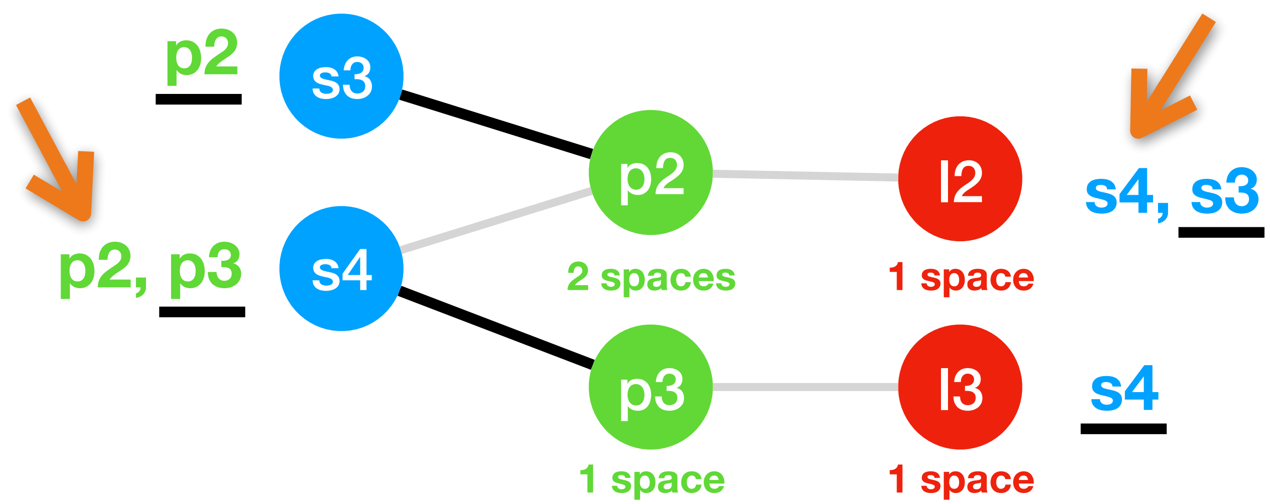

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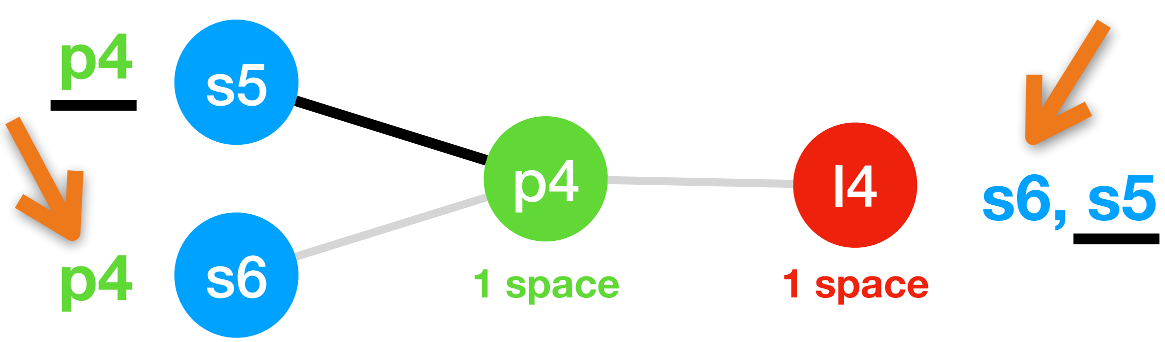

project full, (lecturer full or undersubscribed)

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Two Algorithms for the Student Project Allocation Problem; Journal of Discrete Algorithms; 2007; Abraham, Irving, Manlove

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- **Ties in preference lists** - find a stable matching in polynomial time - but stable matchings are **different sizes**
- Finding a maximum sized stable matching is **NP-hard**.

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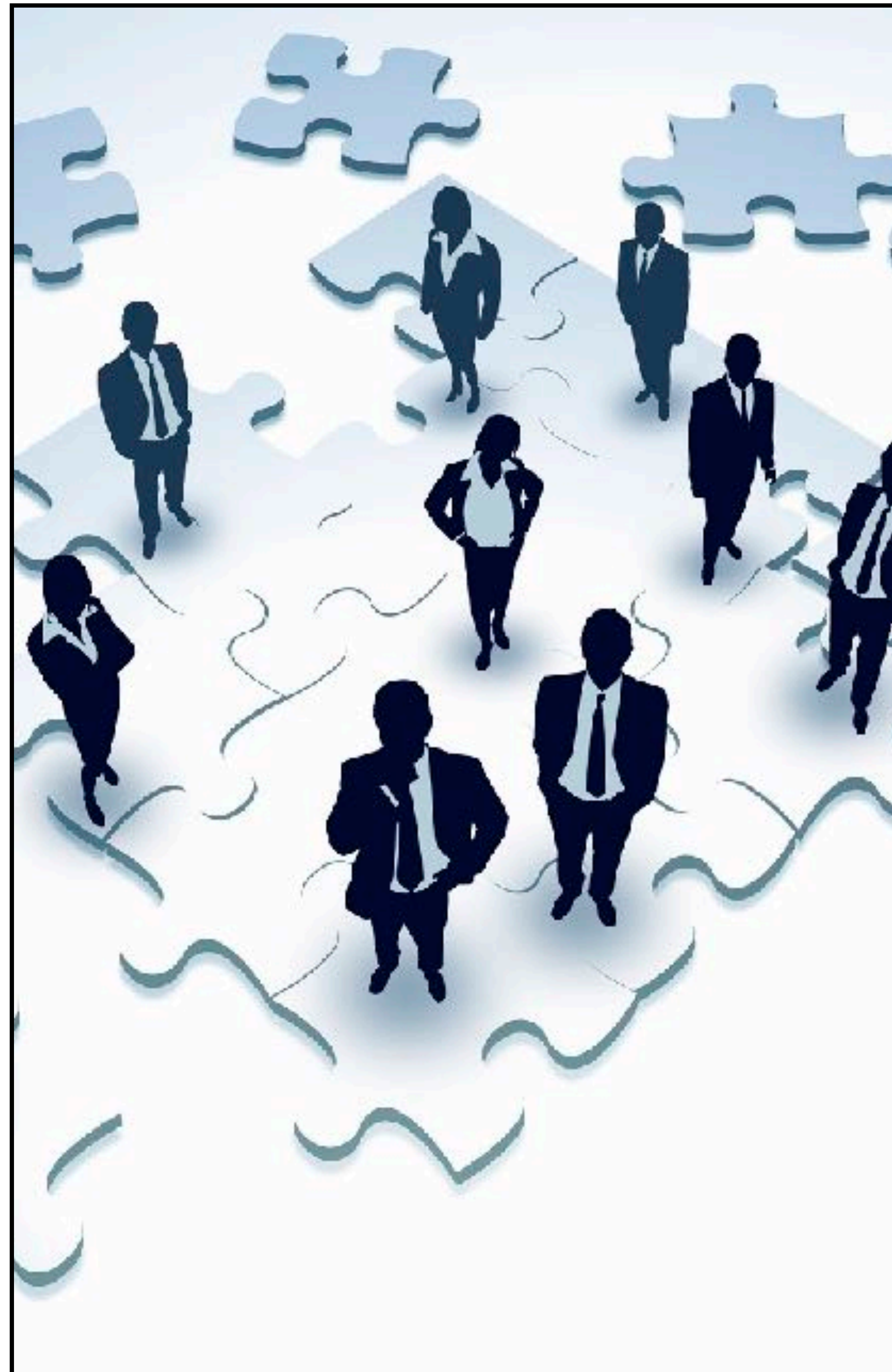
Finding a maximum sized stable matching



Finding a maximum sized stable matching

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Finding a maximum sized stable matching

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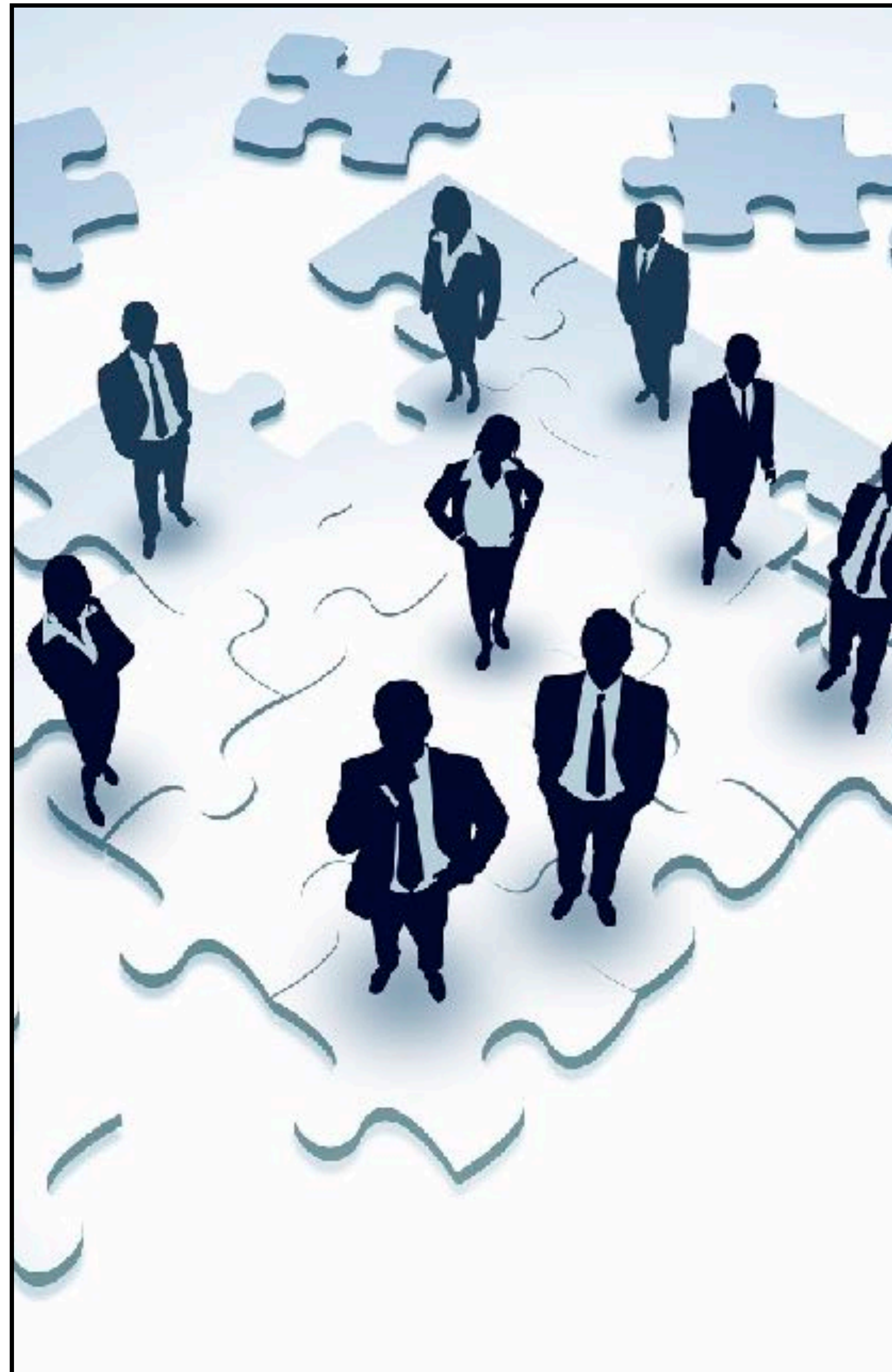
1. Approximation algorithm

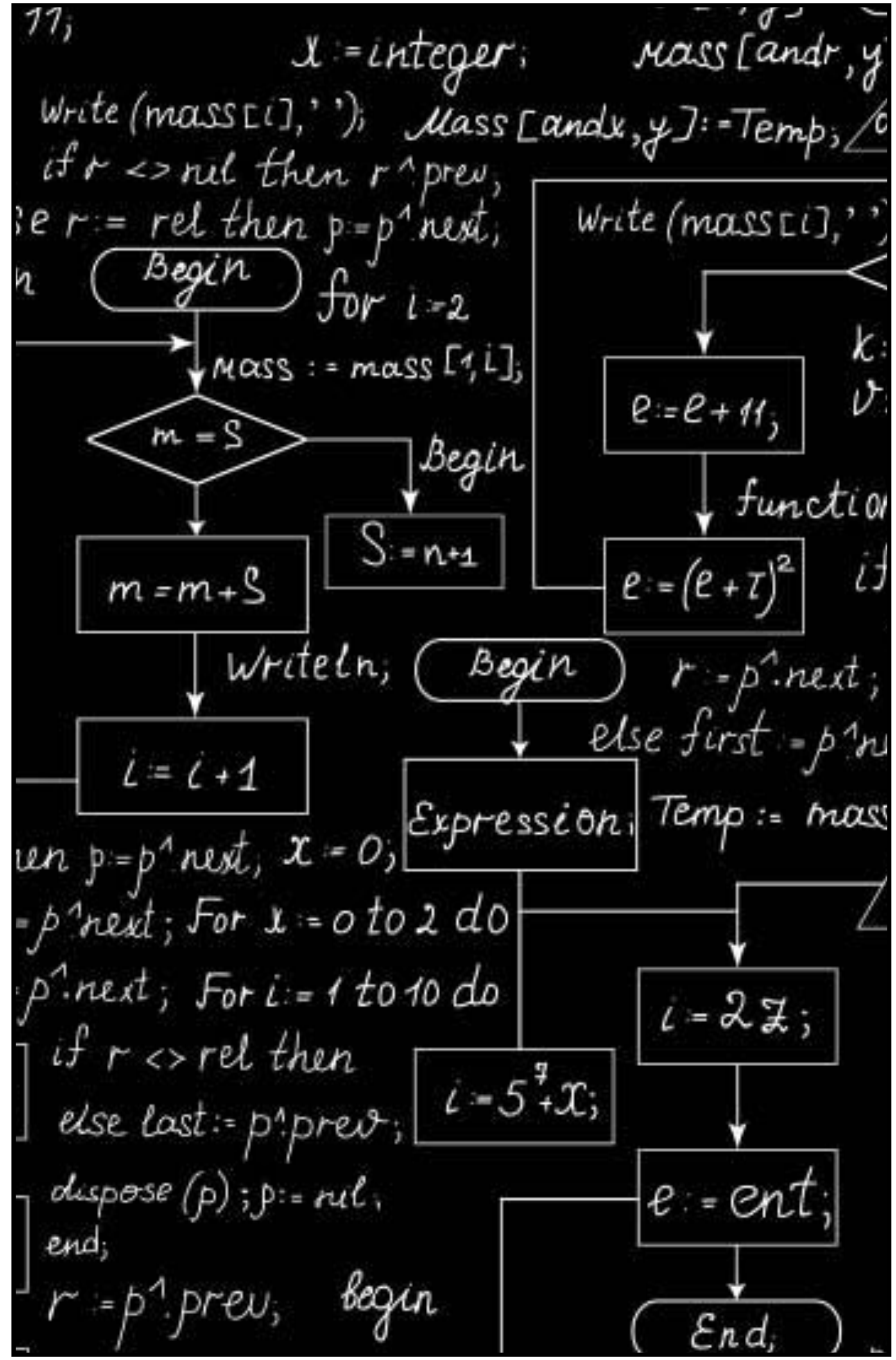


Finding a maximum sized stable matching

Two techniques:

1. Approximation algorithm
2. Integer Programming

Approximation Algorithm



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- Not using a conversion process we tried.

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    - Definition of a blocking pair is more complicated

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- Students iterate twice through their preference list

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Performance guarantee - creating G'

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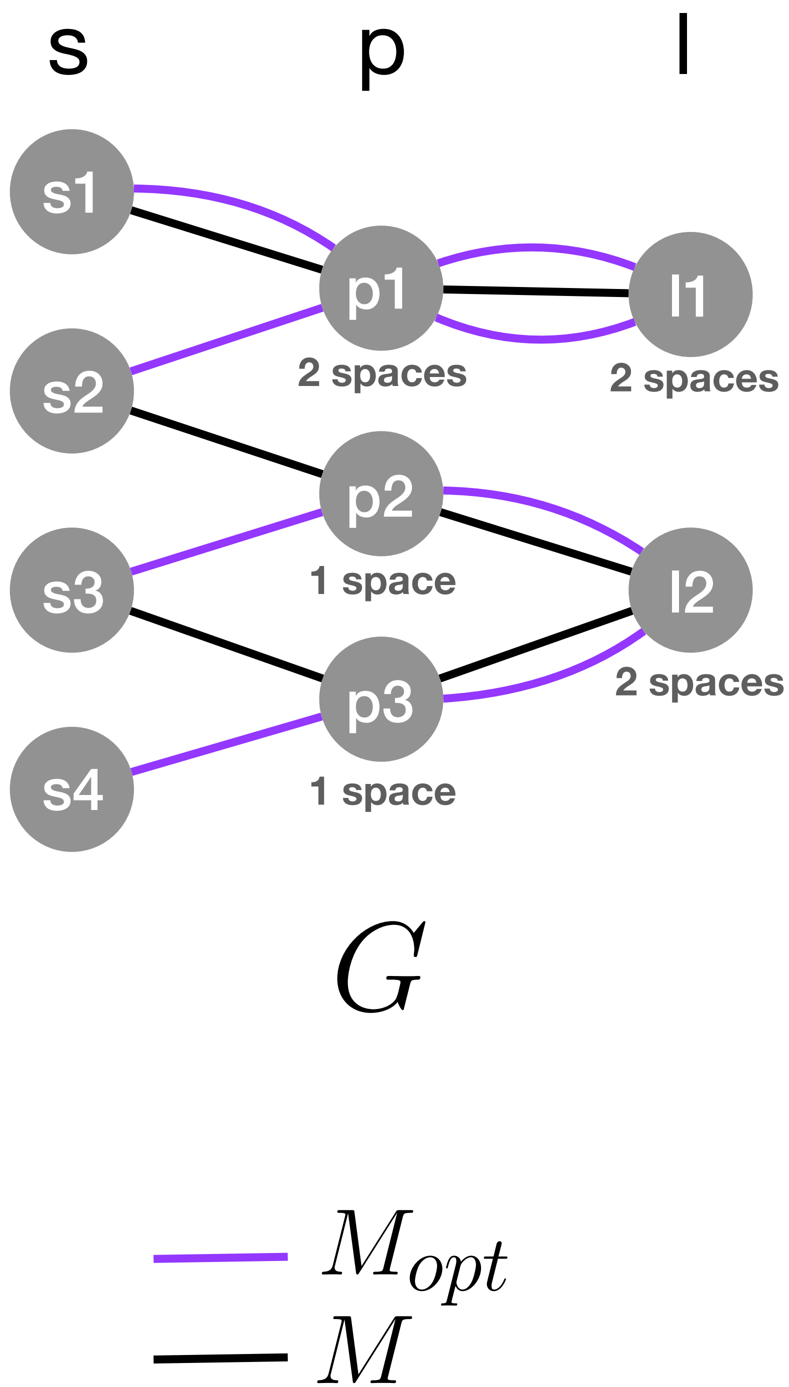

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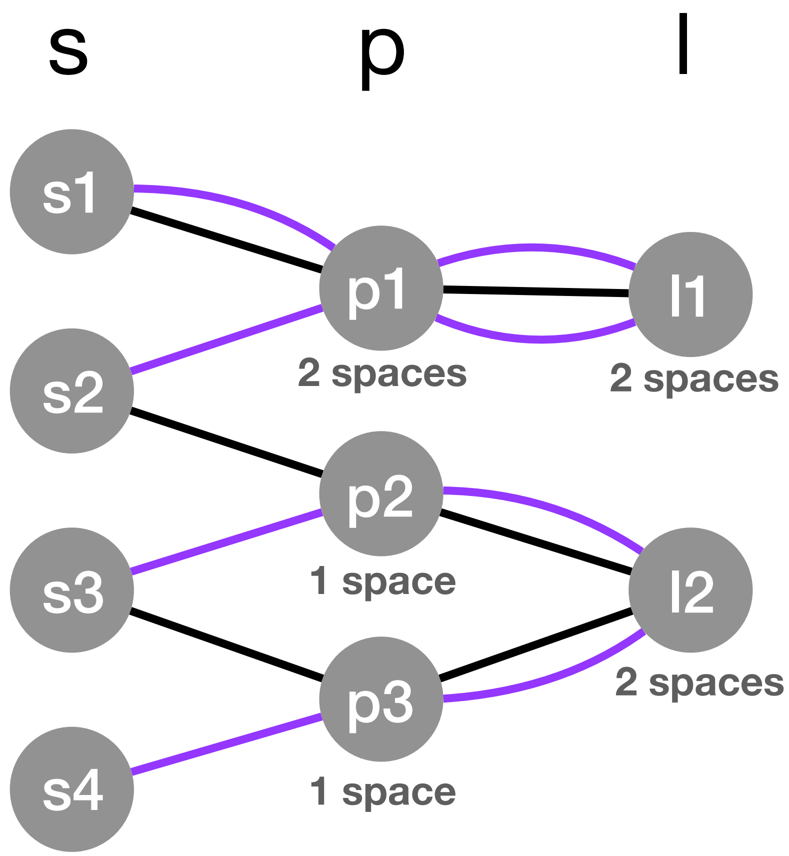

G

G'

M_{opt}

M

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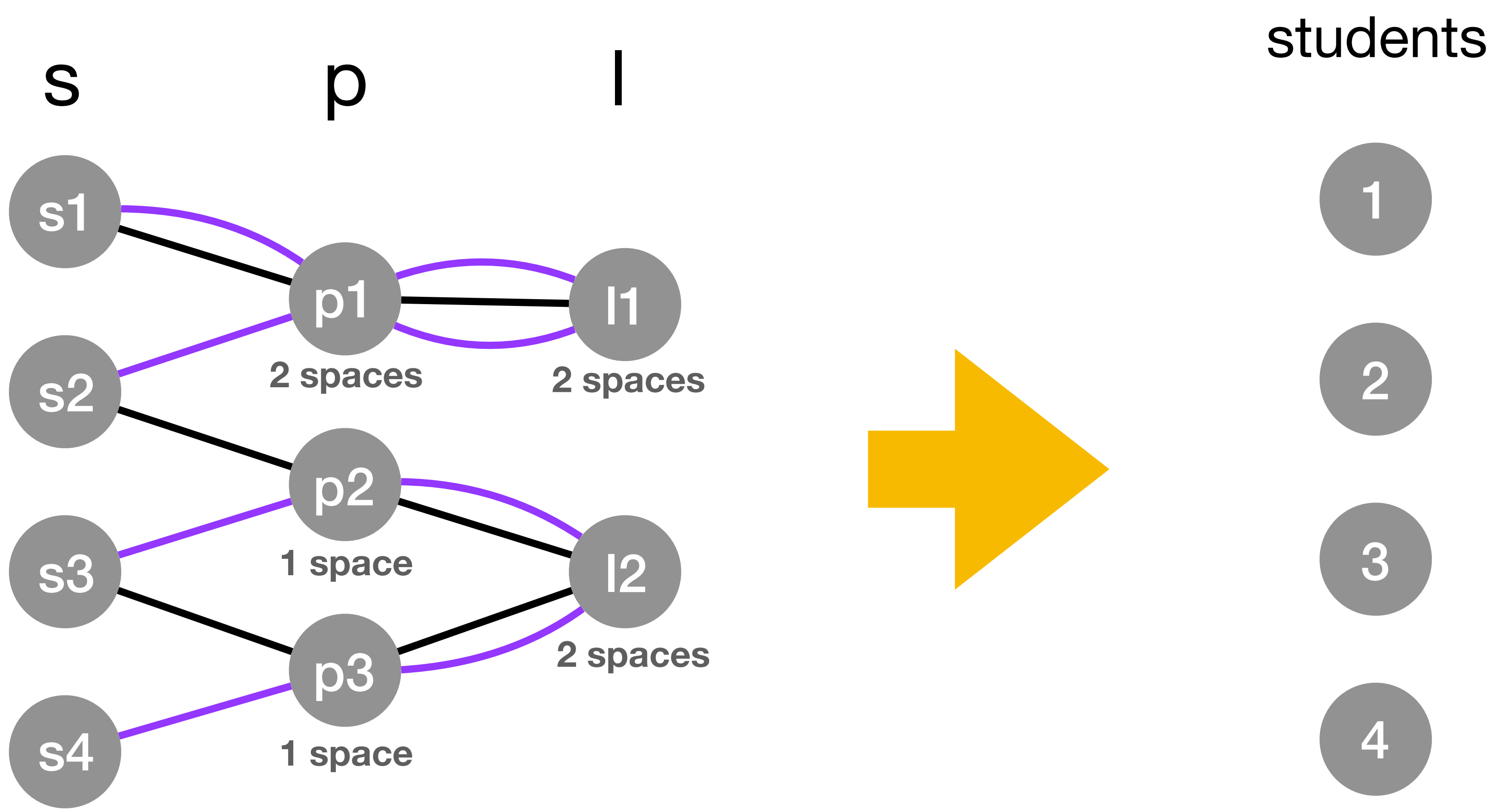

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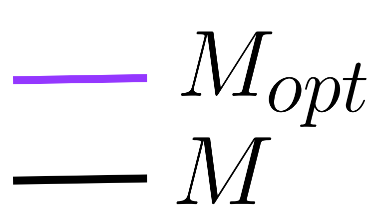

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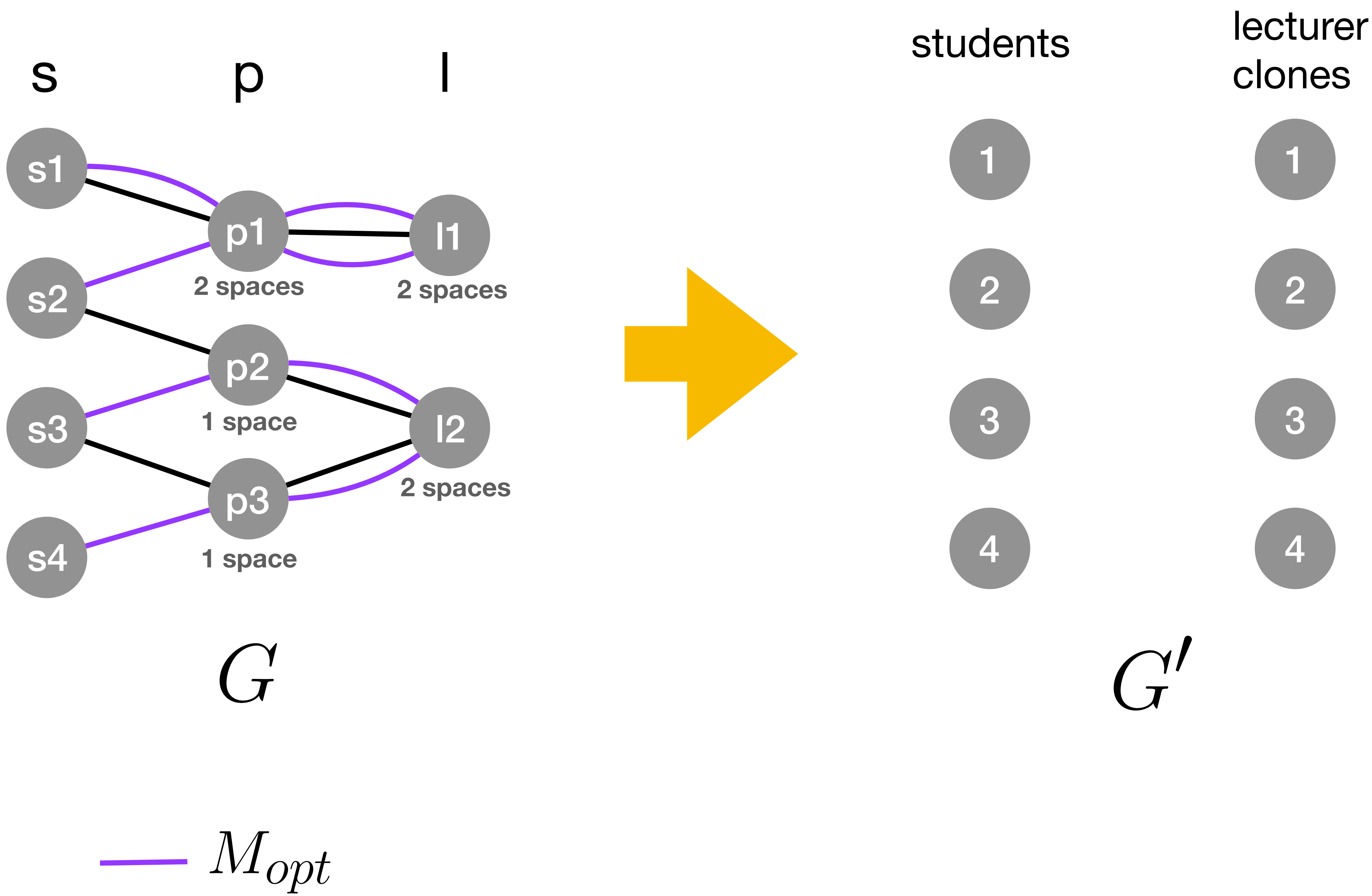

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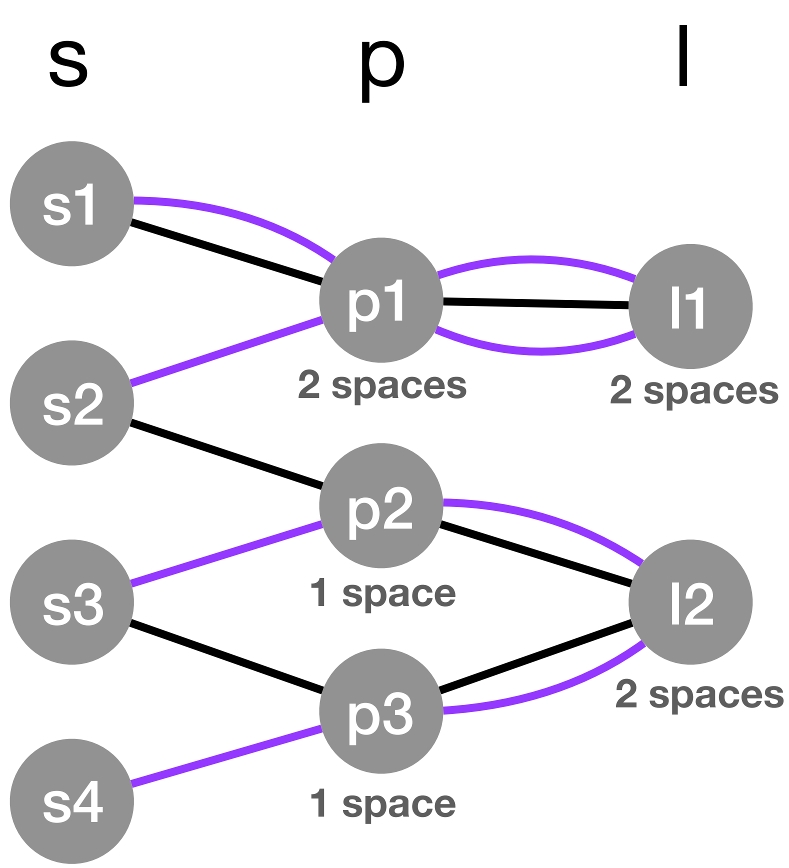
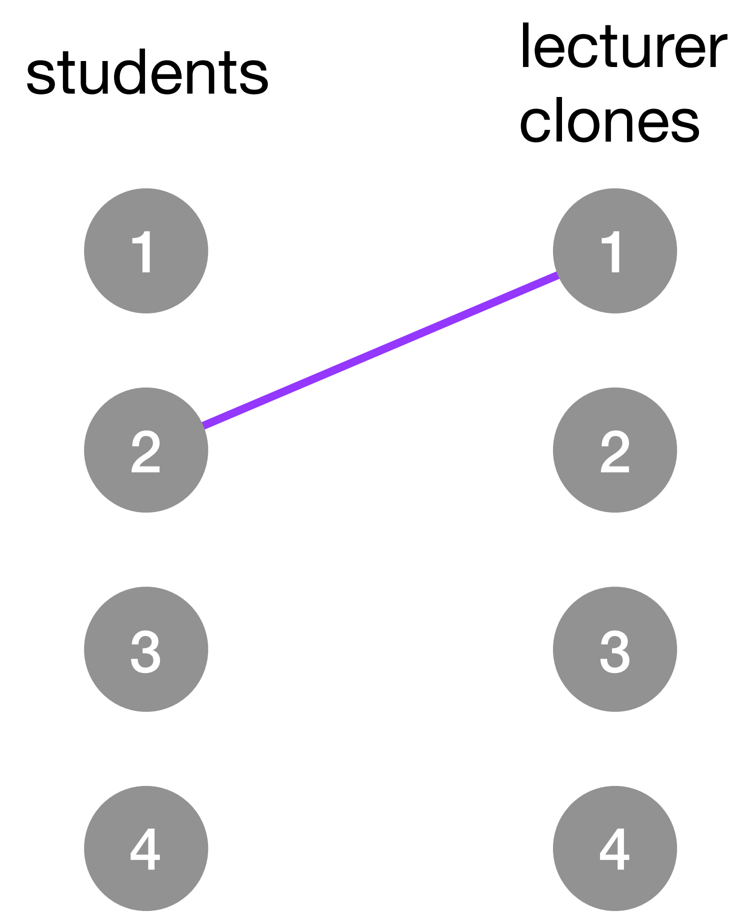
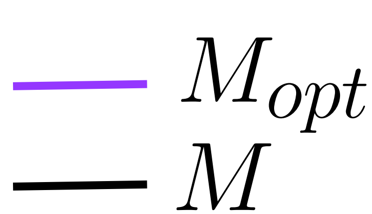


G



G'



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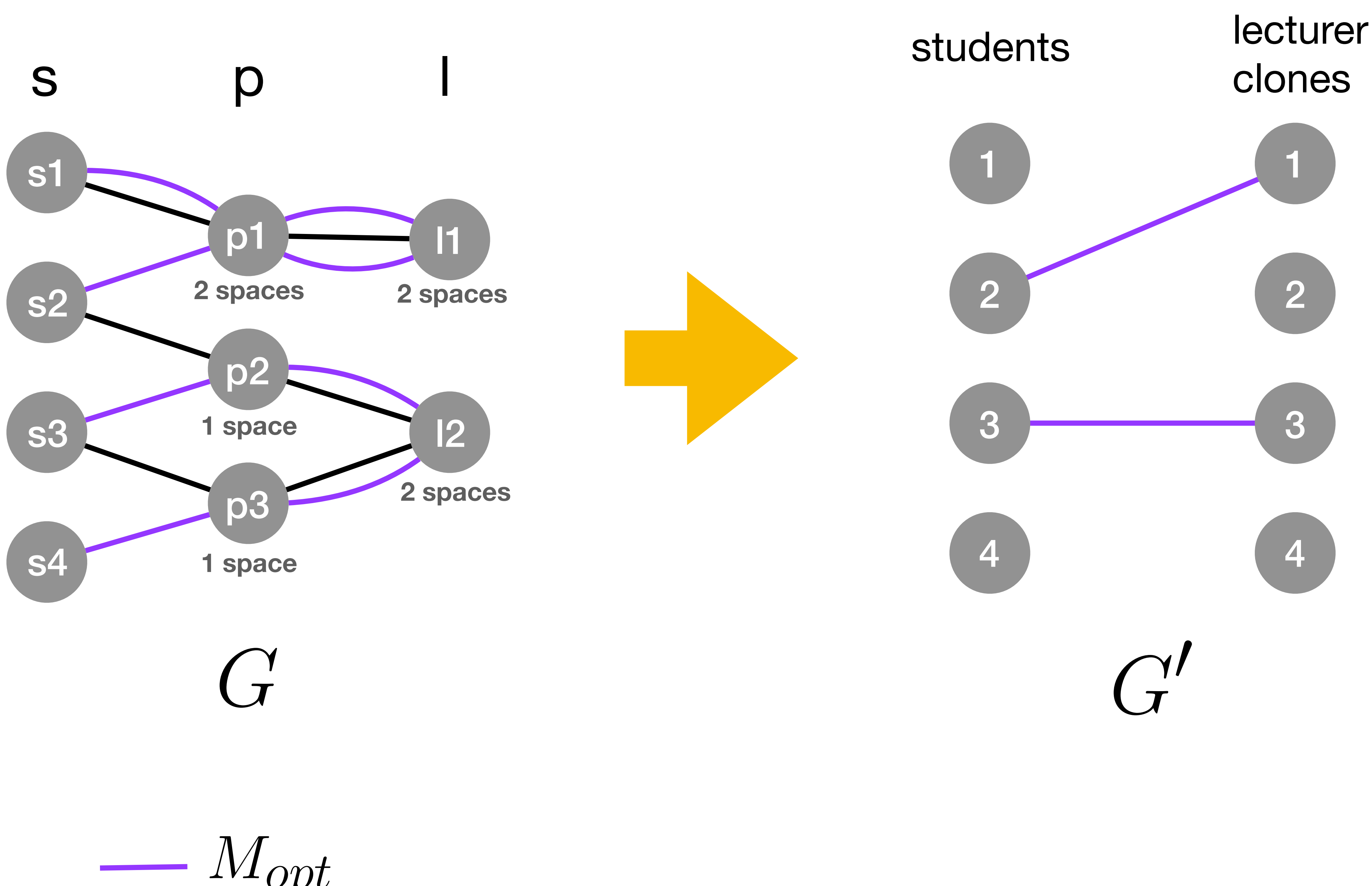

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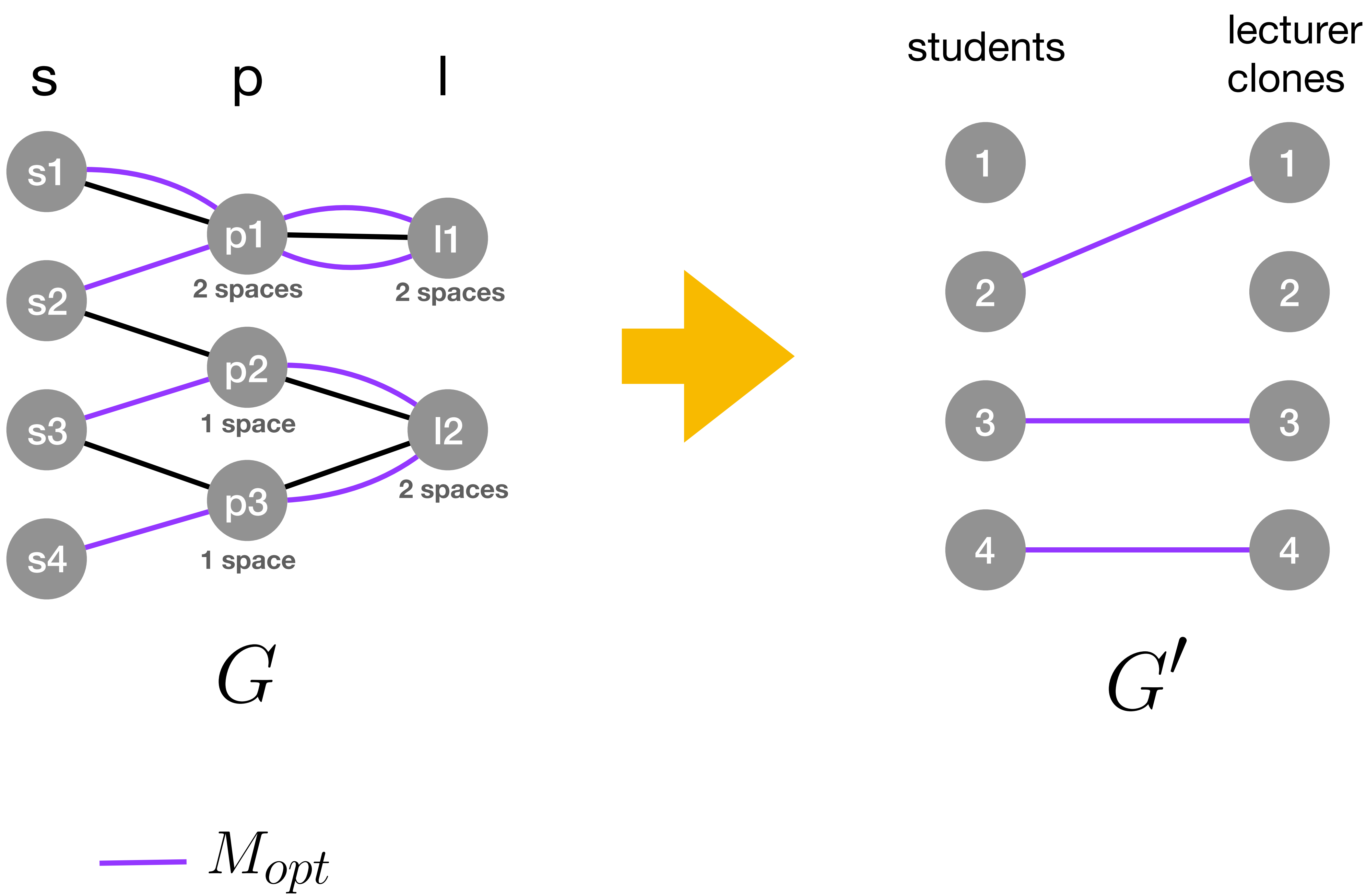

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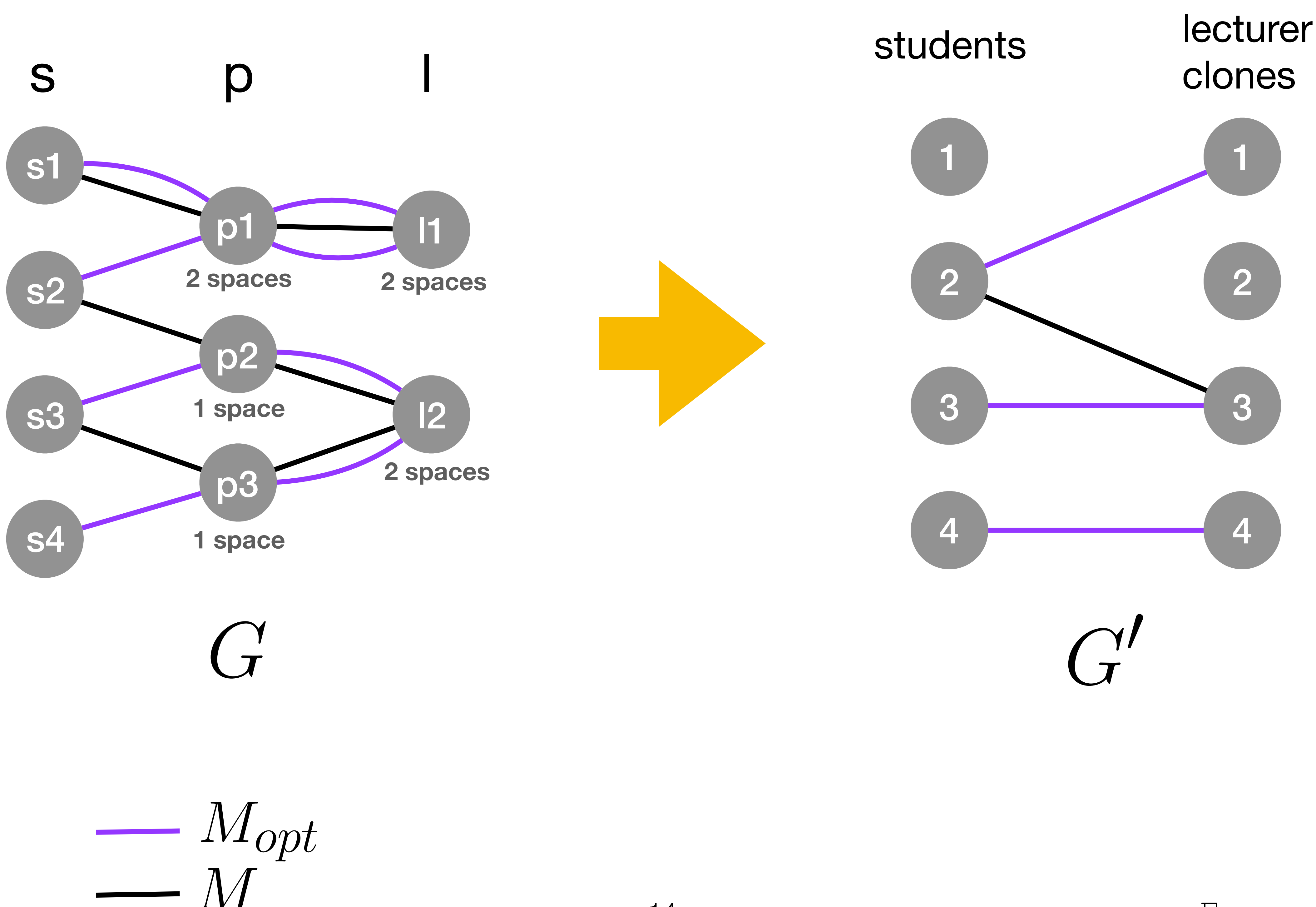

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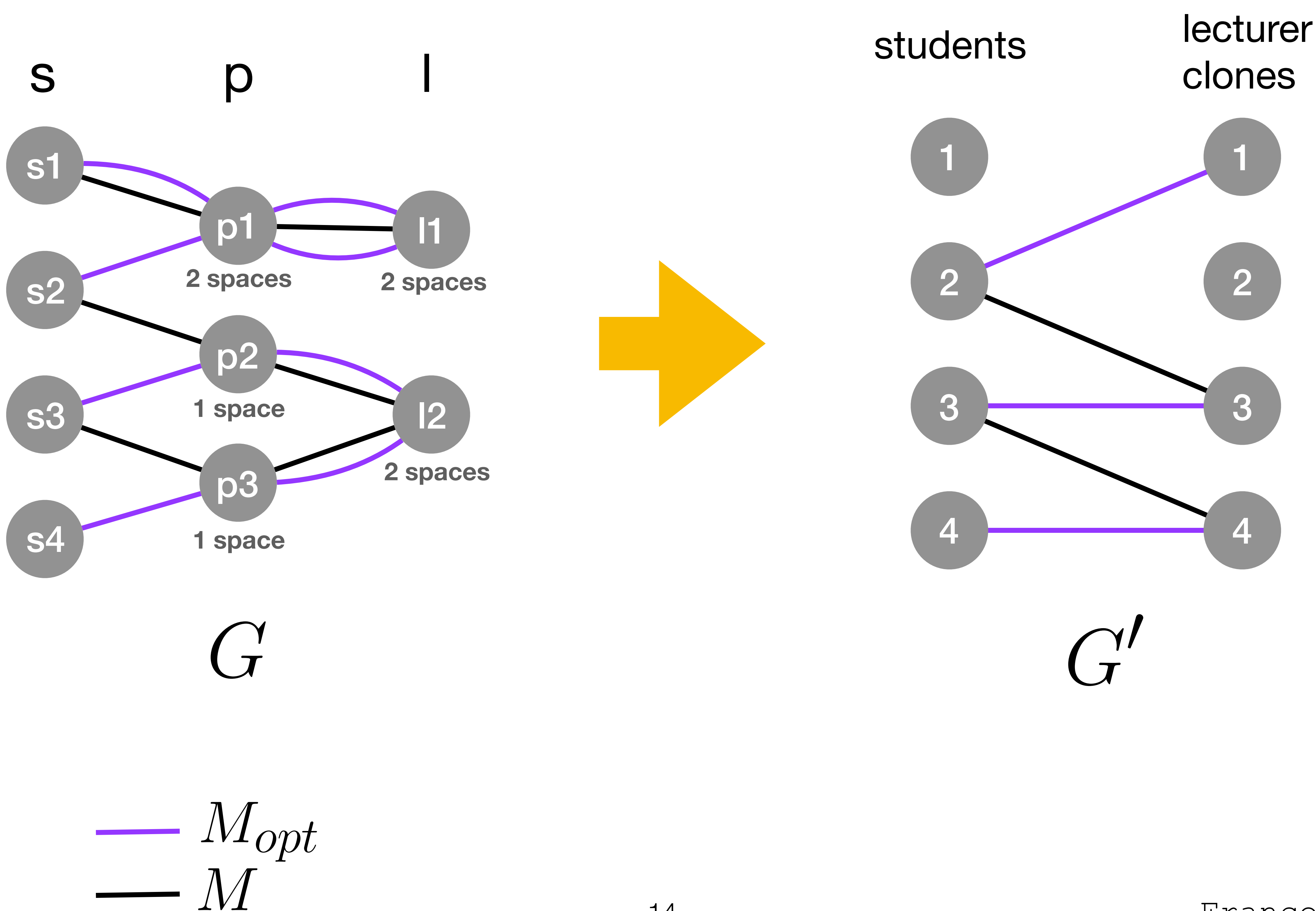

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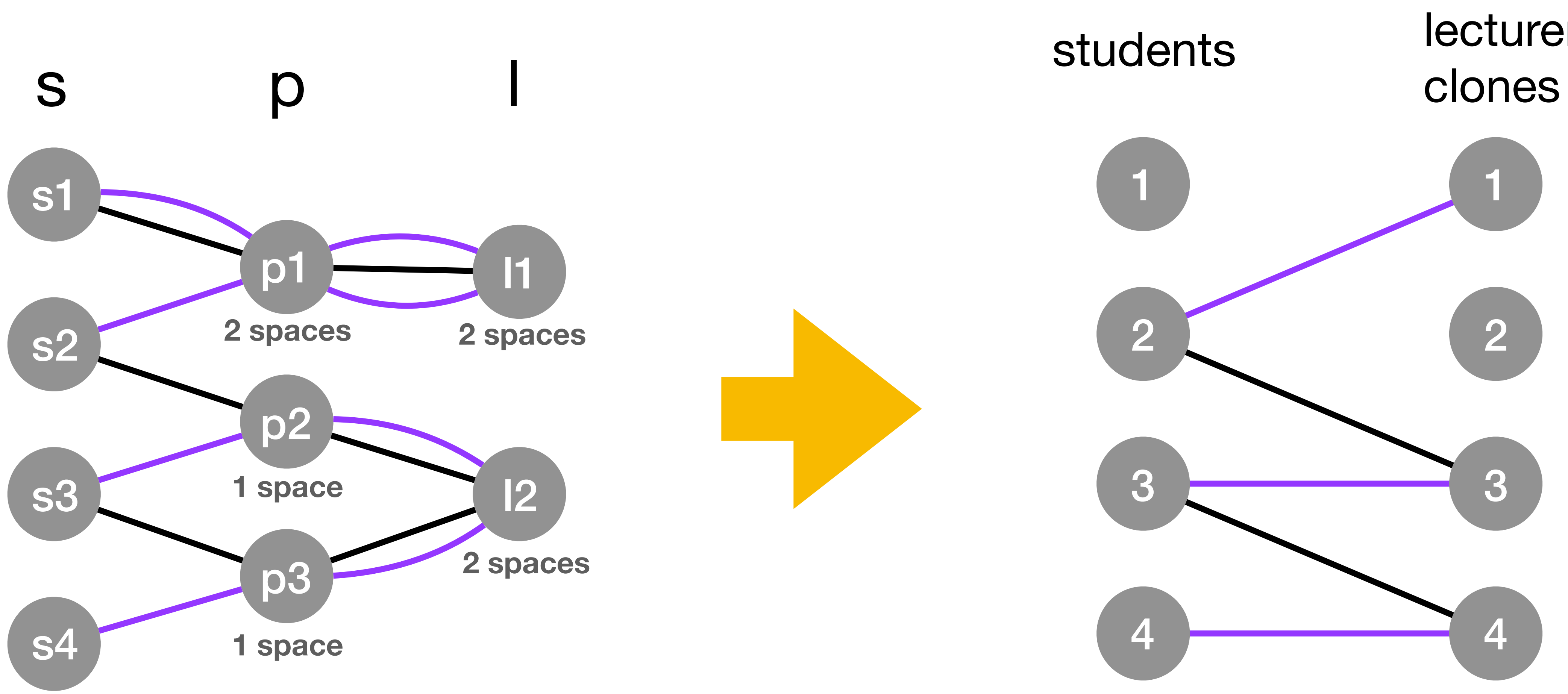

G G'

M_{opt}

M

M'_{opt}

M'

Structures in G'

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- odd length alternating path with end edges in M'_{opt} (number of edges is 3)

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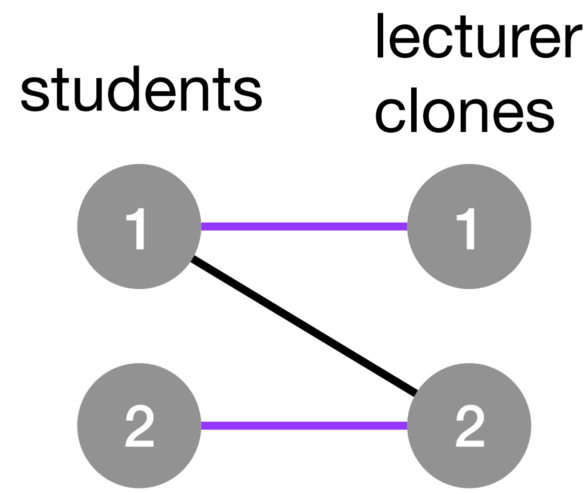

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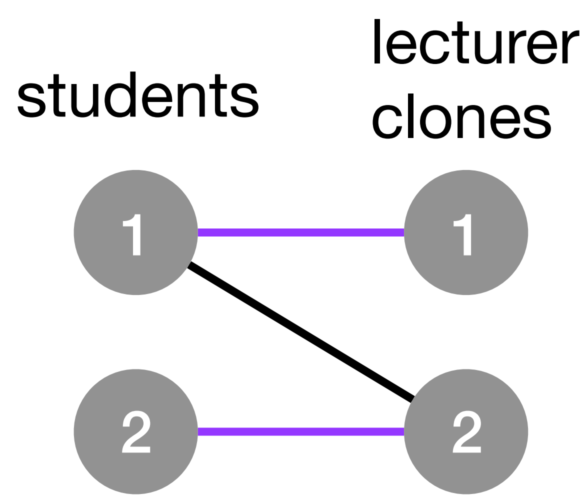

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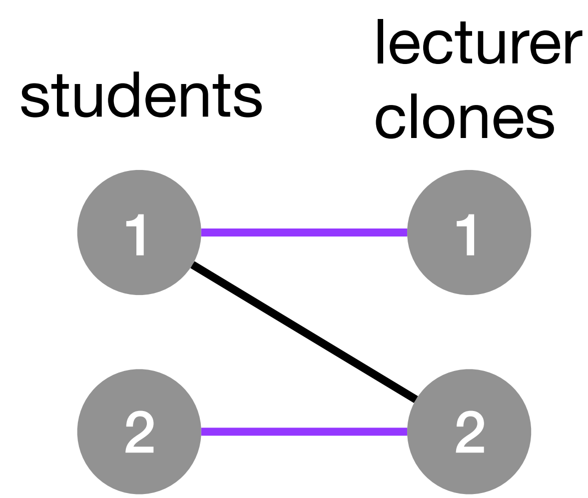

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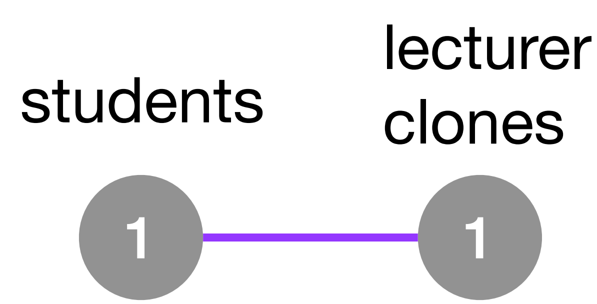

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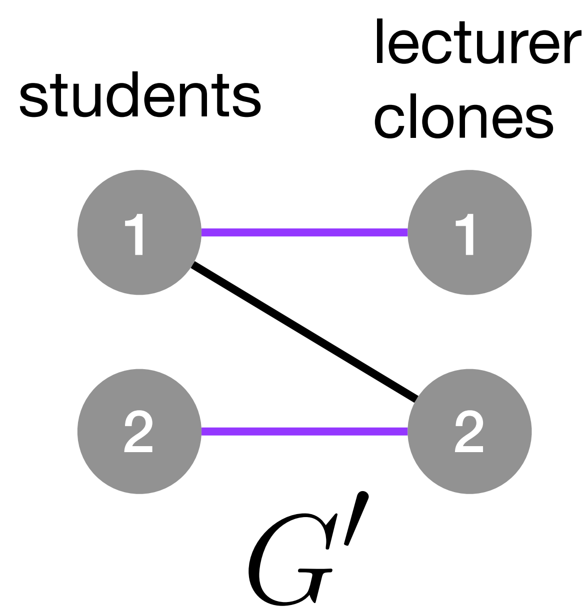

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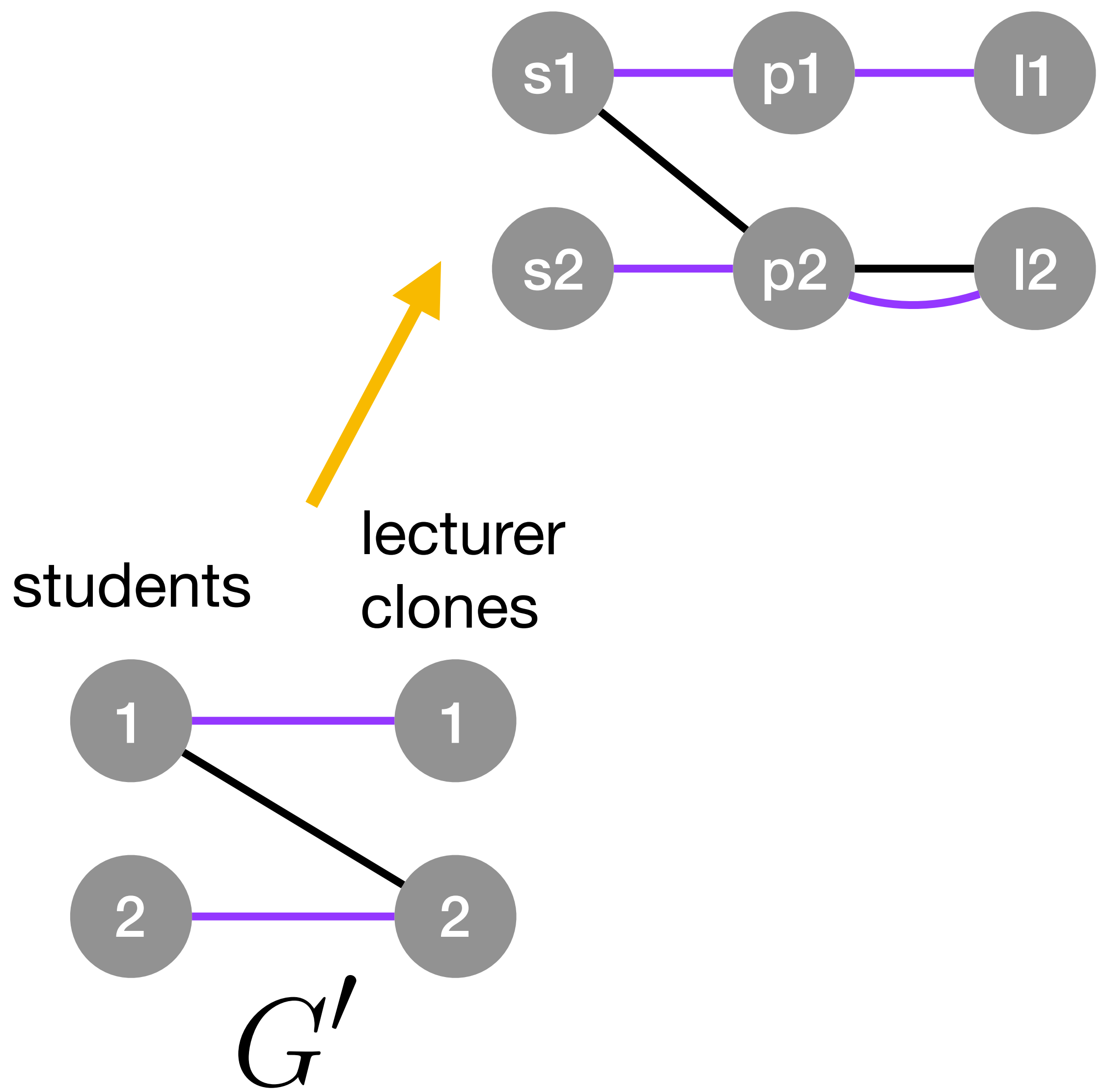

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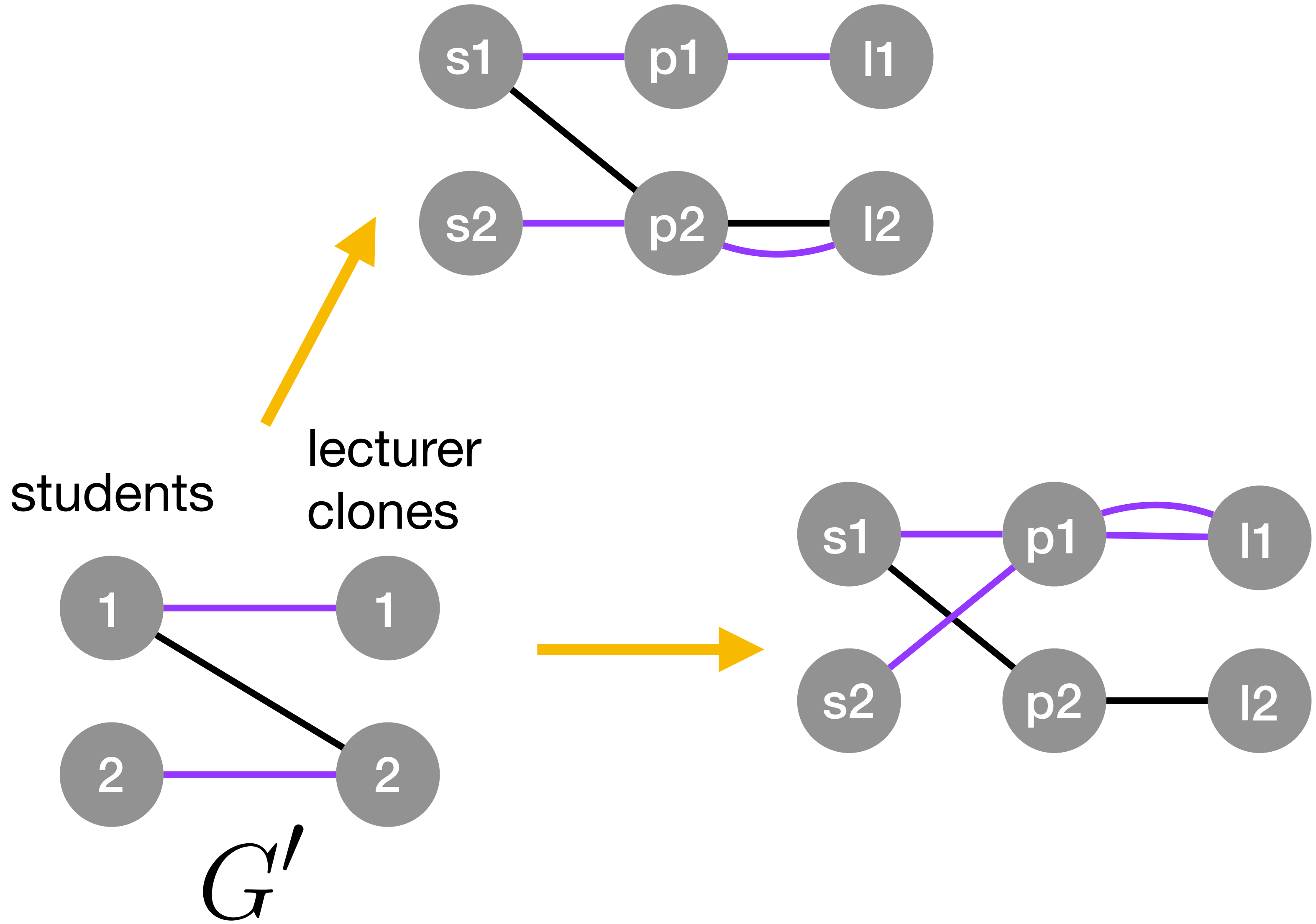

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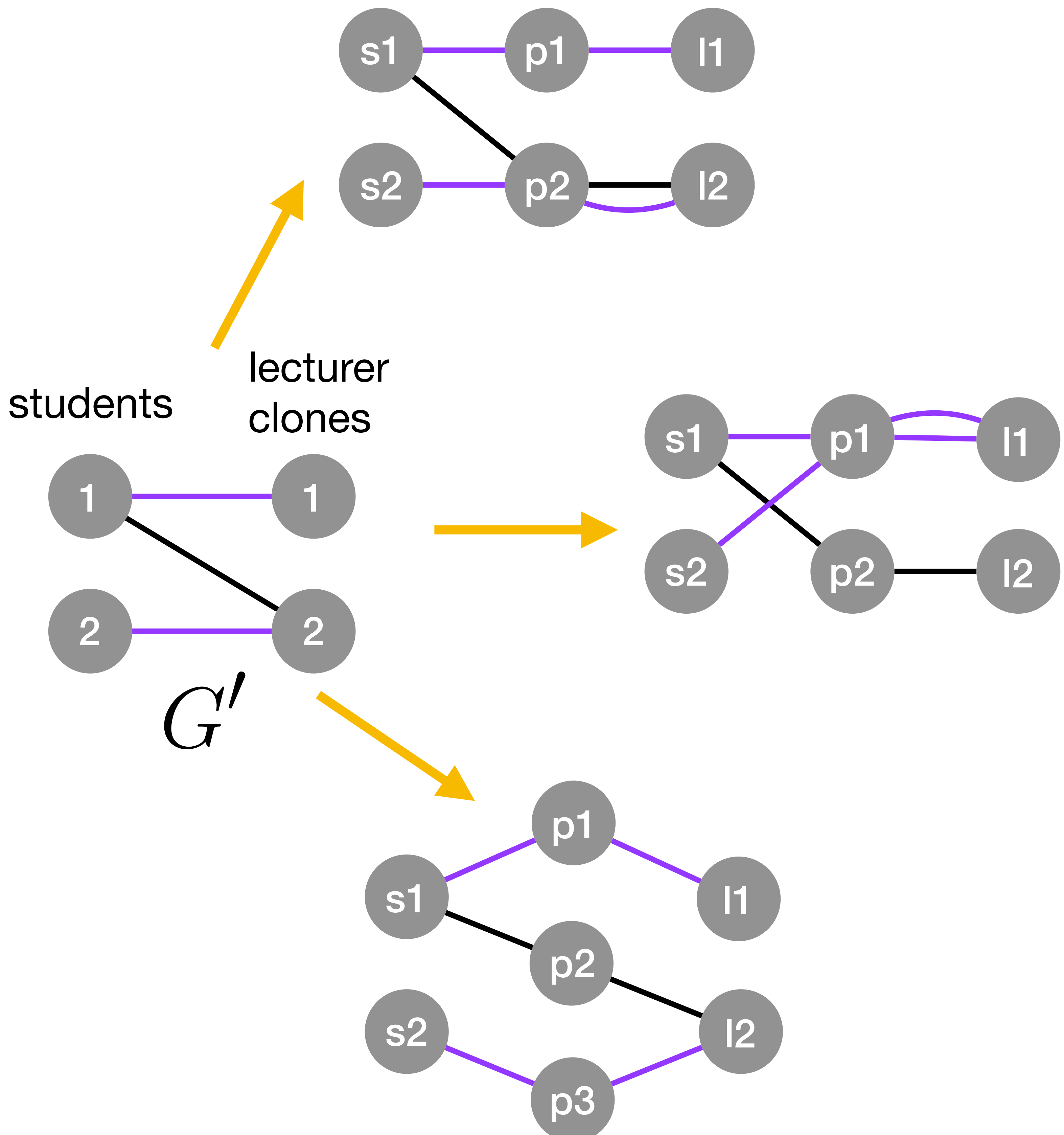

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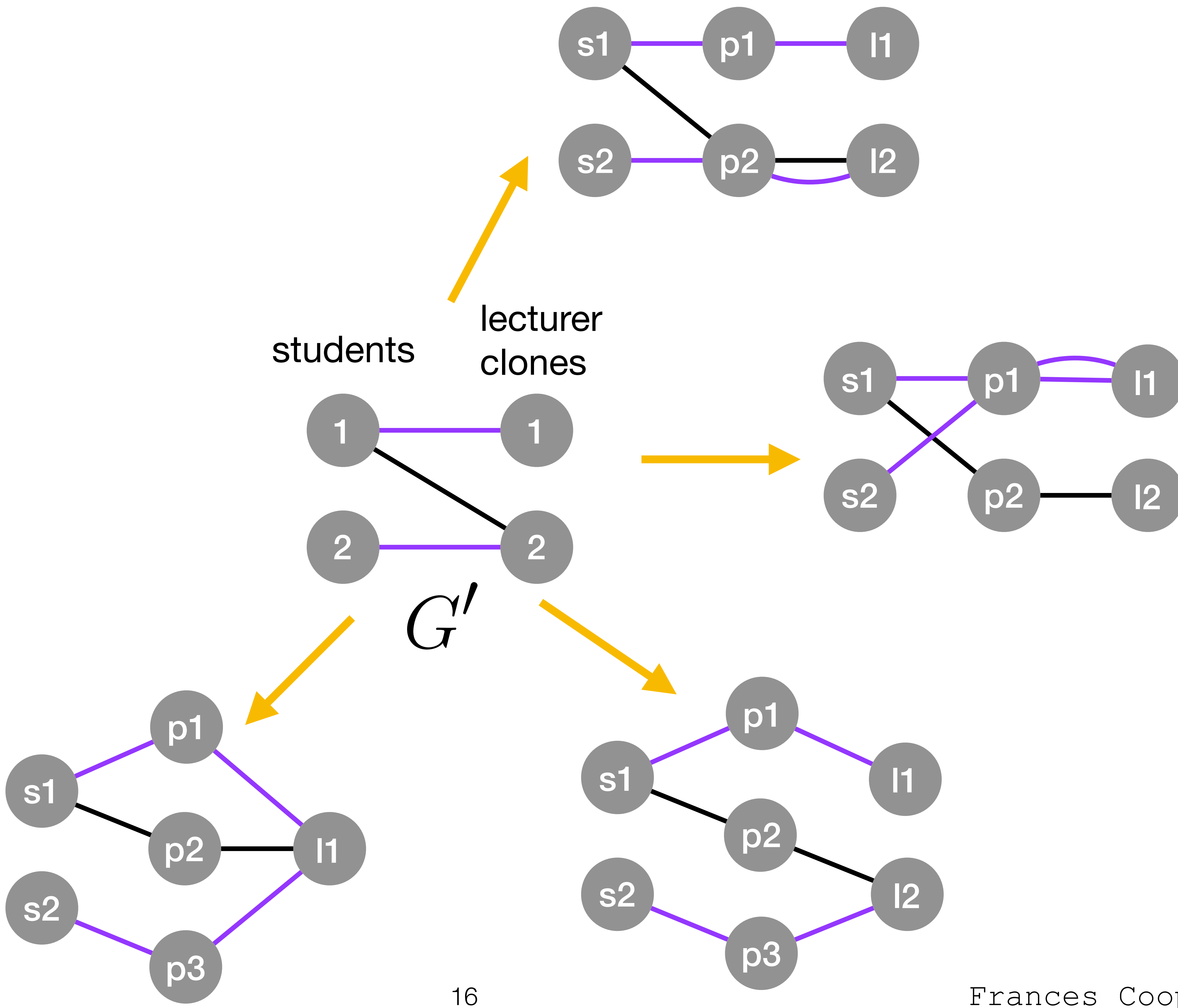

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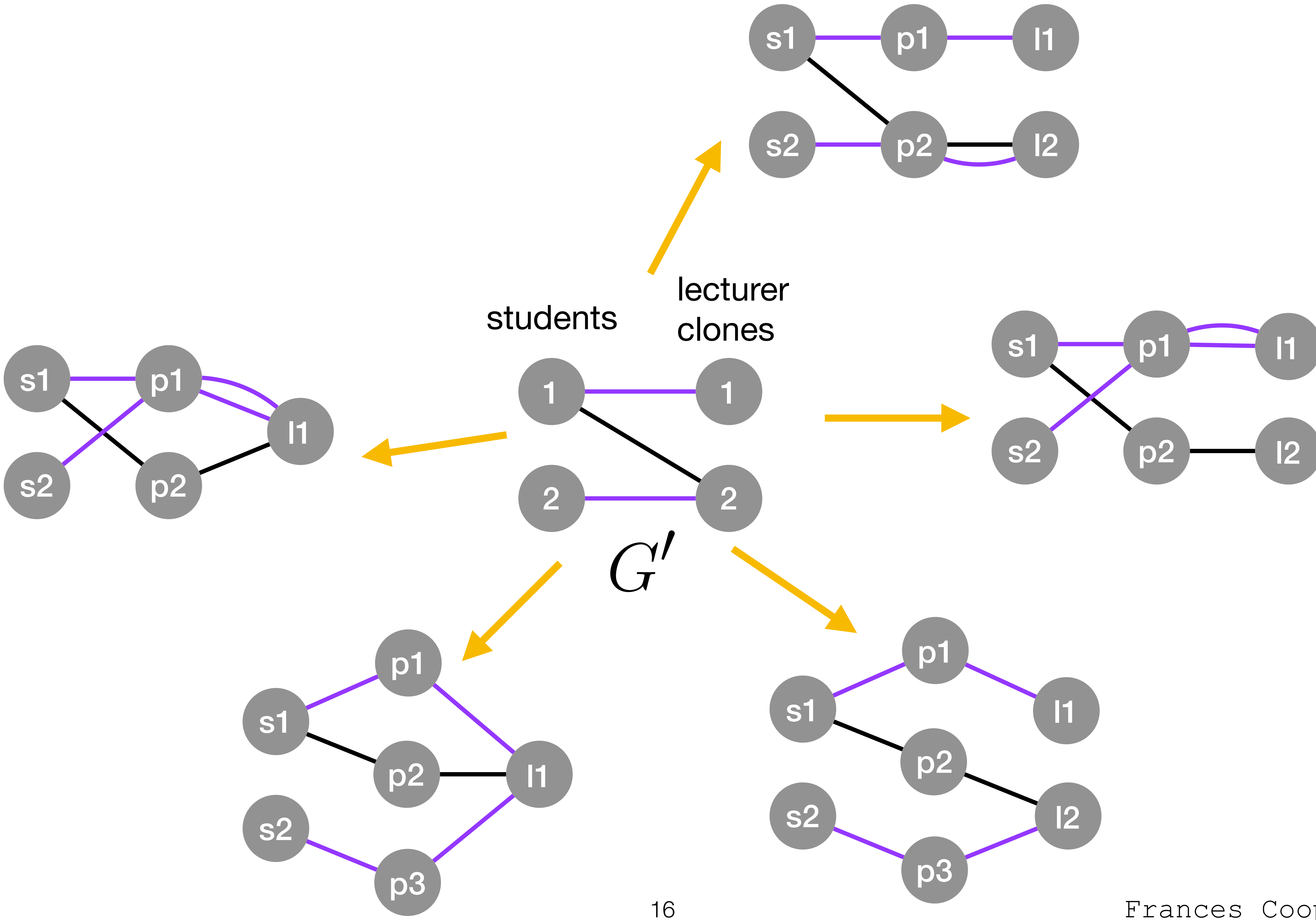

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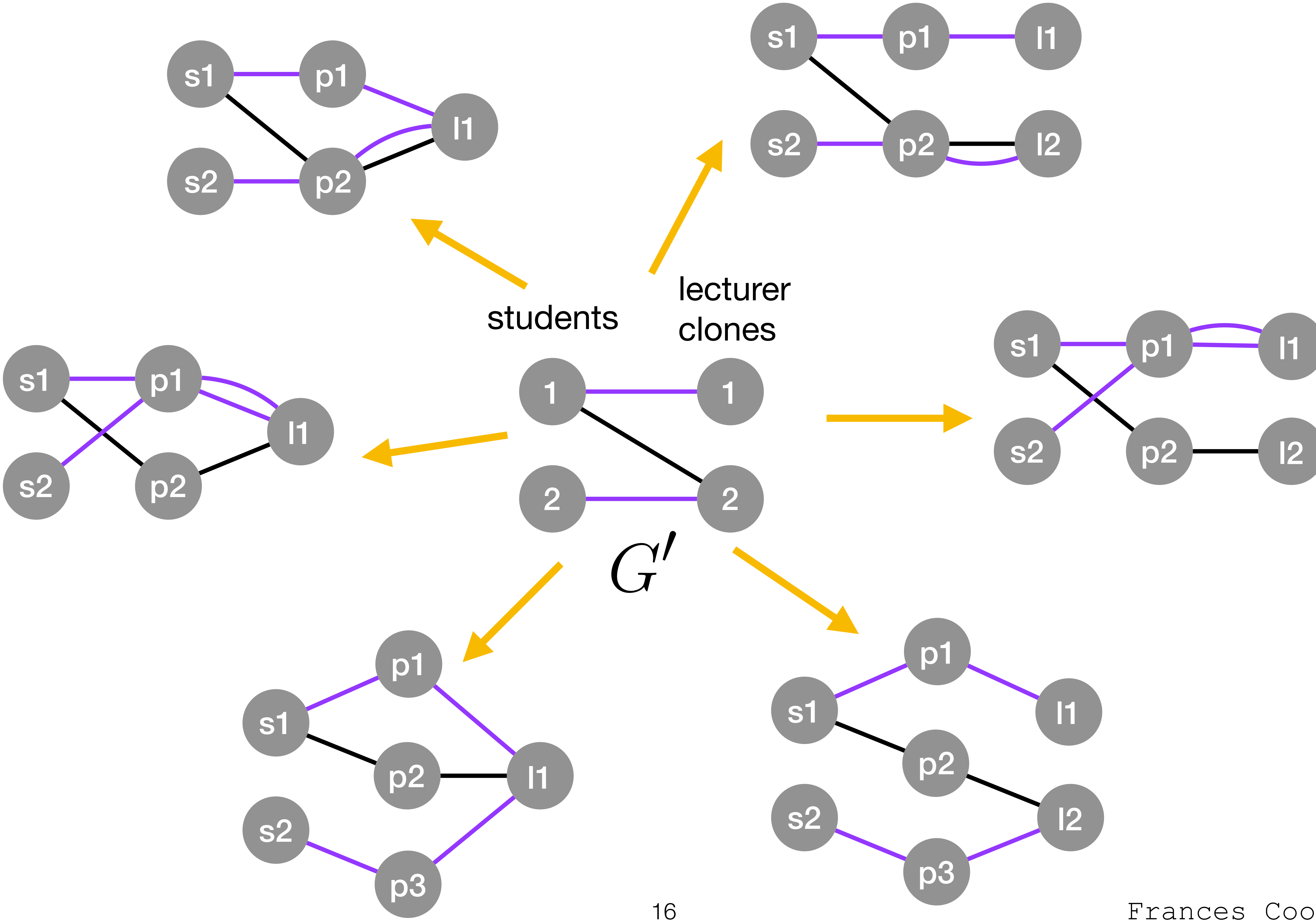

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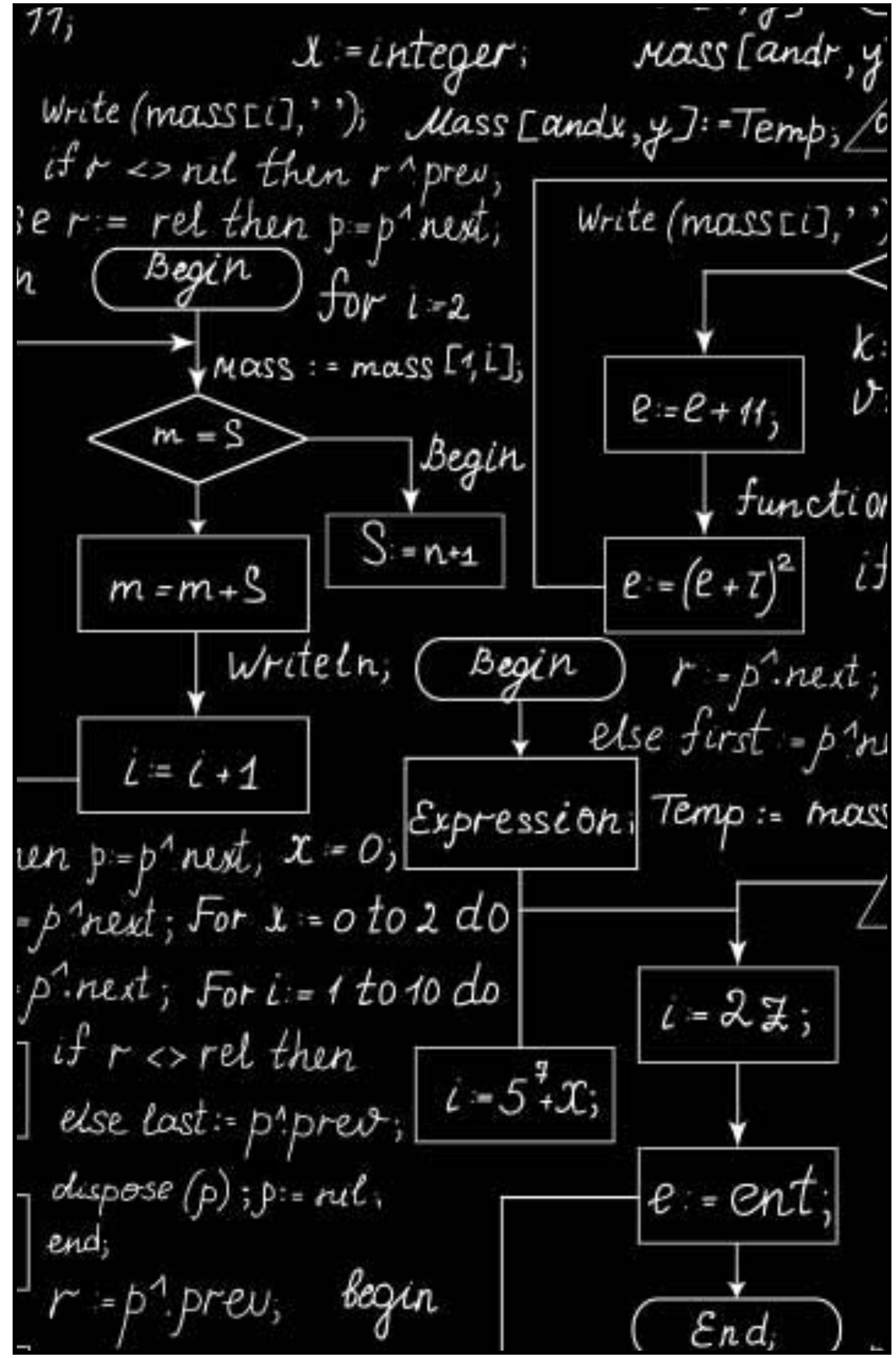
Integer Program



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- gives motivation for using approximation algorithm

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Case	minimum A/Max	average size A/Max	Min/Max
TIES1	1.0000	1.000	1.000
TIES2	0.9792	0.997	0.987
TIES3	0.9722	0.993	0.972
TIES4	0.9655	0.990	0.958
TIES5	0.9626	0.986	0.942
TIES6	0.9558	0.984	0.927
TIES7	0.9486	0.982	0.911
TIES8	0.9527	0.980	0.896
TIES9	0.9467	0.980	0.880
TIES10	0.9529	0.982	0.866
TIES11	0.9467	0.984	0.851

- TIES - 10,000 instances per set, 300 students, 250 projects (capacity 420), 120 lecturers (capacity 360), pref lists length 3 to 5.

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- increasing prob of student and lecturer ties from 0 to 0.5 in 0.05 steps

Experimental Results

- Java (and Gurobi), 100s of thousands of instances with varying parameters. Ran on approximation algorithm and integer program.

- Does the approximation algorithm stick to 2/3 the size of optimal? Or do we get close to maximum?

	minimum	average size	
Case	A/Max	A/Max	Min/Max
TIES1	1.0000	1.000	1.000
TIES2	0.9792	0.997	0.987
TIES3	0.9722	0.993	0.972
TIES4	0.9655	0.990	0.958
TIES5	0.9626	0.986	0.942
TIES6	0.9558	0.984	0.927
TIES7	0.9486	0.982	0.911
TIES8	0.9527	0.980	0.896
TIES9	0.9467	0.980	0.880
TIES10	0.9529	0.982	0.866
TIES11	0.9467	0.984	0.851

- TIES - 10,000 instances per set, 300 students, 250 projects (capacity 420), 120 lecturers (capacity 360), pref lists length 3 to 5.

- increasing prob of student and lecturer ties from 0 to 0.5 in 0.05 steps

- Average approx solution closer to optimal than minimum in all cases

Experimental Results

Experimental Results

Scalability

	instances completed		average total time (ms)	
Case	A	Max	A	Max
SCALS1	10	10	1393.8	227764.3
SCALS2	10	9	5356.7	1096045.6
SCALS3	10	0	13095.3	N/A
SCALS4	10	0	18883.5	N/A
SCALS5	10	0	20993.0	N/A
SCALP1	10	9	193.3	94242.9
SCALP2	10	10	189.4	631225.2
SCALP3	10	3	196.6	882251.0
SCALP4	10	1	248.5	1594201.0
SCALP5	10	0	283.7	N/A
SCALP6	10	0	288.4	N/A

Experimental Results

Scalability

- SCALS - 10,000 students up to 50,000 students. Pref lists 3 to 5 and ties 0.2

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- SCALP - 500 students, ties 0.4, Pref lists increased from 25 to 150 in steps of 25.

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Experimental Results

Scalability

- SCALS - 10,000 students up to 50,000 students. Pref lists 3 to 5 and ties 0.2
- SCALP - 500 students, ties 0.4, Pref lists increased from 25 to 150 in steps of 25.
- much faster than using the integer program

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Experimental Results

- So is it worth using?
- Coram - assigning adopted children to families. ~ 100's of agents. Preference lists long and probability of ties high
- 21 instances, increasing difficulty. Initial IP could only solve first 6 within 5 minutes, approximation algorithm took less than 2 seconds for each

Future Work

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- Finding an approximation algorithm with a better performance guarantee than 3/2

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- Finding a better inapproximability result than 33/29

Approximation Algorithms for Stable Matching Problems; PhD thesis; 2007; H. Yanagisawa

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- Finding an approximation algorithm with a better performance guarantee than 3/2
- Finding a better inapproximability result than 33/29
- coalitions:

Approximation Algorithms for Stable Matching Problems; PhD thesis; 2007; H. Yanagisawa

Future Work

- Finding an approximation algorithm with a better performance guarantee than 3/2
- Finding a better inapproximability result than 33/29
- coalitions:
    - group of several students and lecturers

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Future Work

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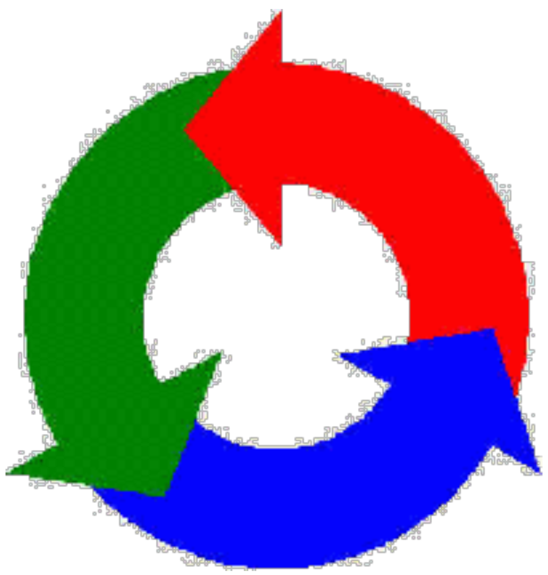
- coalitions:

Approximation Algorithms for Stable Matching Problems; PhD thesis; 2007; H. Yanagisawa

  - group of several students and lecturers

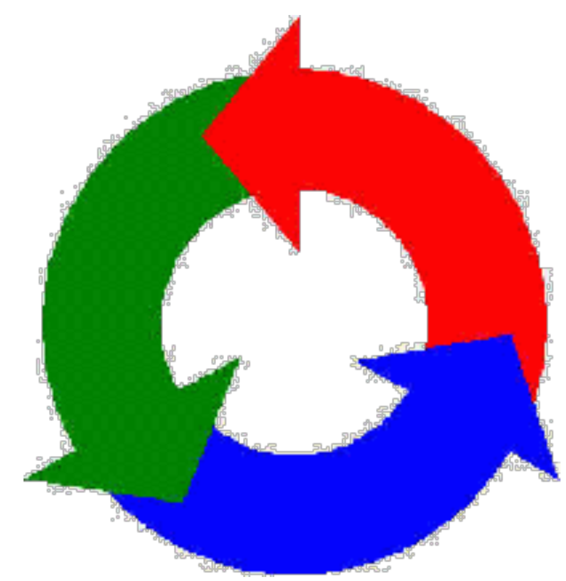
  - permute their assignments



Future Work

- Finding an approximation algorithm with a better performance guarantee than 3/2
- Finding a better inapproximability result than 33/29
- coalitions:
    - group of several students and lecturers
    - permute their assignments
    - some or all get a better outcome

Approximation Algorithms for Stable Matching Problems; PhD thesis; 2007; H. Yanagisawa



Thank you

Summary

- Student-project allocation problem
- Finding a maximum stable matching
  - Integer programming
  - Approximation algorithm
- Future work: improved performance guarantee; improved inapproximability result; coalitions

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EPSRC Doctoral Training Account